



The Open University
Mathematics/Science/Technology
An Inter-faculty Second Level Course
MST204 Mathematical Models and Methods

mathematical models and methods

unit 1 Recurrence relations



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Unit 1

Recurrence relations

Prepared for the Course Team
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The Open University, Walton Hall, Milton Keynes.

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Introduction

We have chosen to begin this course with a unit on recurrence relations because it is a topic we hope you will find fairly straightforward, while at the same time it is of both practical and theoretical value. Its practical value is partly that some mathematical models depend on recurrence relations and partly that numerical methods for solving differential equations depend on recurrence relations. The theoretical value of the topic is that it will introduce you to some mathematical ideas connected with the solution of linear equations, which will be followed up later in the course.

In this unit recurrence relations will be examined in some detail, to see how they work, how to solve them, and how to cope with difficulties which can arise in using them.

Study guide

Sections 1, 3, 4 (television) and 6 (computing) should be studied in that order but Section 2 (tape) can be studied at any stage between Sections 1 and 5. This means that if you find yourself short of time before the TV programme you can leave your study of Section 2 until after the programme. To get the most out of the TV programme in Section 4 you should have studied Sections 1 and 3 beforehand. You can do the Exercises in Section 5 at any stage after studying Sections 1 to 4. (You may prefer to use these for revision later.)

Sections 3 and 4 both involve numerical work using a calculator. The results quoted in this unit were obtained using a calculator which displayed eight figures while working with ten figures. Your calculator will probably give slightly different answers from those in the unit.

It is planned that you should visit a terminal during your work for the first assignment as a means of practising some of the techniques and consolidating some of the ideas in *Units 1* and *2*. The package to be used for your computer work is described in *Unit 2*. When you make your visit to the terminal you should take both *Units 1* and *2* with you.

1 Introduction to recurrence relations

If you have studied *M101* or *MS283* you will have met recurrence relations already under the heading 'formula iteration' and most of Subsection 1.1 may be familiar to you. If you think this is so, we suggest that you read it quickly and do the exercises at the end of the subsection. If you can do them, do not spend any more time on Subsection 1.1.

1.1 What is a recurrence relation?

A familiar question in an intelligence test is to be given a sequence of numbers and to be asked to determine the next number in the sequence. For example if you were given the sequence 1, 3, 9, 27, 81 you would probably be able to deduce that the next **term** (i.e. member) in the sequence is three times the previous term and $3 \times 81 = 243$. The idea used is that the terms in the

sequence are *related* to each other. If we say that u_r represents the r th term in the sequence and if we start with u_0 , the sequence $u_0, u_1, u_2, u_3, \dots$ above can be generated by the iteration formula

$$u_{r+1} = 3u_r \tag{1}$$

where we specify that $u_0 = 1$ is the initial term in the sequence. The formula (1) can be used to determine u_1 as

$$u_1 = 3u_0 = 3 \times 1 = 3.$$

From this we use the formula (1) with $r = 1$, to give

$$u_2 = 3u_1 = 3 \times 3 = 9$$

and so on, generating the sequence 1, 3, 9, 27, 81, 243, ...

The formula

$$u_{r+1} = 3u_r$$

is called a **recurrence relation** because it is a *relation* between succeeding members of the sequence which can be used *recurrently* to generate the sequence once the initial term is known. It is this idea that we want to explore in this unit as it is fundamental to many of the methods for solving problems on the calculator or the computer.

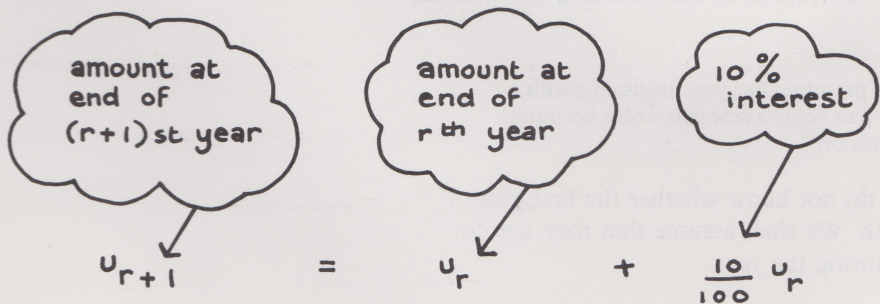
The following examples illustrate some problems which give rise to recurrence relations.

Example 1

Consider the problem of determining how much money you will have in your deposit account after five years if you start with £1000 and the interest rate is 10% compounded annually. (Assume that you make no deposits or withdrawals in this period.)

After one year your £1000 will have produced an extra £100 in interest so that you now have £1100 in the account. In the second year you receive £110 interest and the account increases to £1210. How do we express this problem mathematically?

If we let u_r represent the amount in the account after r years then we can write down u_{r+1} , the amount at the end of the following year, as



i.e. $u_{r+1} = 1.1u_r.$

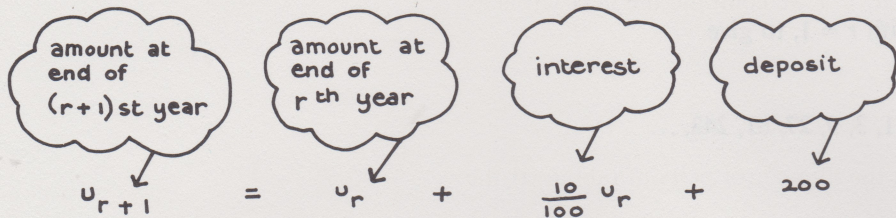
We start, at the beginning of the first year, with £1000 so that we write $u_0 = 1000$. The calculator works very well on such problems, giving the following sequence.

r	0	1	2	3	4	5
u_r	1000	1100	1210	1331	1464.10	1610.51

After five years your account would have increased to £1610.51.

Example 2

A more complicated banking problem would be one in which you start with £1000 in your deposit account and deposit £200 at the end of each subsequent year. If the interest remains at 10% throughout, the recurrence relation representing the growth of the account is



i.e.

$$u_{r+1} = 1.1u_r + 200$$

The following sequence was computed with the help of a calculator.

r	0	1	2	3	4	5
u_r	1000	1300	1630	1993	2392.30	2831.53

Your account, after five years, totals £2831.53.

(In Section 4 recurrence relations will be applied to the similar but more realistic problem of mortgage repayments.)

So far we have only seen the case in which the recurrence relation gives u_{r+1} in terms of u_r . In the next example u_{r+1} depends on the previous *two* terms.

Example 3

In 1202 an Italian merchant, Leonardo of Pisa, known to us as Fibonacci published a book on algebra, *Liber Abaci* ('Book of the Abacus'), in which he posed the following problem:

'How many pairs of rabbits will be produced in a year, beginning with a single pair, if in every month each pair bears a new pair which becomes productive from the second month on?'

The question is a little unclear as we do not know whether the first pair of rabbits is productive in the first month. We shall assume that they are not. We also assume that no rabbits die during the year.

We let u_r denote the number of pairs at the end of month r . Since there is one pair of rabbits at the beginning of the year we have $u_0 = 1$. Since we assume that this pair is non-productive there are no new rabbits in the first month and $u_1 = 1$.

To set up the recurrence relation we need some way of determining the number of pairs at the end of month $(r + 1)$. Since we have assumed that no rabbits die we can write the equation

$$\begin{bmatrix} \text{number of pairs} \\ \text{after } r + 1 \text{ months} \end{bmatrix} = \begin{bmatrix} \text{number of pairs} \\ \text{after } r \text{ months} \end{bmatrix} + \begin{bmatrix} \text{number of new pairs born} \\ \text{in the } (r + 1)\text{st month} \end{bmatrix}$$

i.e. $u_{r+1} = u_r + \begin{bmatrix} \text{number of new pairs born} \\ \text{in the } (r + 1)\text{st month} \end{bmatrix}$

However, we know how many pairs were born in this month because each productive pair produces one new pair. The question tells us that the number of productive pairs are those pairs which are over one month old. Only the pairs of rabbits that were alive at the end of the $(r - 1)$ st month satisfy this condition and there are u_{r-1} of these. Hence the required recurrence relation is

$$u_{r+1} = u_r + u_{r-1}.$$

Using the known values $u_0 = 1$ and $u_1 = 1$ we compute the next terms as

$$\begin{aligned} u_2 &= u_1 + u_0 = 1 + 1 = 2 \\ u_3 &= u_2 + u_1 = 2 + 1 = 3 \\ u_4 &= u_3 + u_2 = 3 + 2 = 5 \end{aligned}$$

and so on. The following table gives the number of pairs of rabbits at the end of each month.

Month: r	0	1	2	3	4	5	6	7	8	9	10	11	12
No. of pairs. u_r	1	1	2	3	5	8	13	21	34	55	89	144	233

The solution to Fibonacci's problem of determining how many rabbits there are at the end of the year is 233 pairs.

The numbers in the above table are called the Fibonacci numbers. These numbers have some remarkable properties which have fascinated mathematicians over the years. In fact there is a current journal, the *Fibonacci Quarterly*, which is devoted to their study. We shall meet them again in Section 2.

Example 4

(Do not spend much time on this example as we shall not refer to it again.)
If you have studied MS283 or M101 you will have met problems such as: determine the largest root of the cubic equation

$$x^3 - 4x^2 + x + 1 = 0.$$

Dividing through by x^2 and taking all but the first term over to the right hand side of the equation gives

$$x = 4 - \frac{(x + 1)}{x^2}.$$

One method of attempting to locate the largest root is to use the recurrence relation

$$x_{r+1} = 4 - \frac{(x_r + 1)}{x_r^2}$$

and to iterate in the hope that a point will be reached at which $x_{r+1} \simeq x_r$

(i.e. x_{r+1} is approximately equal to x_r).

The symbol $\simeq$ means 'approximately equal to'.

If this happens an approximation to the root of the original equation has been found.

Starting with $x_0 = 3.5$ the following sequence was obtained using the calculator.

r	0	1	2	3	4	5	6	7
x_r	3.5	3.6326531	3.6489395	3.6508432	3.6510643	3.6510900	3.6510930	3.6510934

Clearly this is converging to a limit, and we should be quite happy to say that the root is approximately $x = 3.65109$ to five decimal places. Why this sequence of numbers converges, why it converges to the largest root and to what accuracy the result can be quoted are questions which do not directly concern us here. The purpose of the example is simply to illustrate one of the ways in which recurrence relations can arise.

In Examples 1, 2 and 4 each of the recurrence relations was of the form

$$u_{r+1} = f(u_r) \quad (r = 0, 1, 2, \dots)$$

or
$$x_{r+1} = f(x_r) \quad (r = 0, 1, 2, \dots)$$

where f is a given function. It makes no difference whether we express the recurrence relation in terms of x_r or u_r since it is the relationship between the terms that we are interested in. Yet another way of writing the same recurrence relation is

$$u_r = f(u_{r-1}) \quad (r = 1, 2, 3, \dots).$$

We are interested in the relationship itself, not the way it is written down.

The initial term that is needed to specify the sequence is called the **initial condition**. In Examples 1 and 2 we were given a value for u_0 and this initial condition enables us to compute as many terms in the sequence as we wish. There are other possibilities: you will frequently see, for example, an initial condition given for u_1 . This again is acceptable as we are still able to generate the sequence $u_1, u_2, u_3, u_4 \dots$ using the value for u_1 .

The Fibonacci example is a little more complicated since each member of the sequence depends on the previous two and we needed two initial conditions to start the calculation. A general recurrence relation of this type is

$$u_{r+1} = f(u_r, u_{r-1})$$

where $f(u_r, u_{r-1})$ stands for some given formula containing one or both of the variables u_r and u_{r-1} . (In Example 3 we had $f(u_r, u_{r-1}) = u_r + u_{r-1}$.)

It is also possible to have recurrence relations in which the formula for u_{r+1} involves three or more of the preceding members of the sequence. This still satisfies the requirement that the terms in the sequence are related and that the formula can be used recurrently to generate the successive terms in the sequence.

Finally, the formula giving the recurrence relation may depend on r as well as the previous terms. An example is the recurrence relation

$$u_{r+1} = ru_r$$

Exercise 1

Which of the following are recurrence relations?

(i) $u_{r+1} = \frac{1}{2}(u_r + u_{r-1})$

(ii) $u_2 = 3u_1 - 5$

(iii) $x_{r+1} = \frac{1}{2}\left(x_r + \frac{4}{x_r}\right)$

(iv) $u_r = \frac{1}{(r+4)}u_{r-1}$

(v) $u_r = 3u_{r+1} - 2u_{r+2}$

(Solution on p. 49).

Exercise 2

In the following exercises, compute the next five terms.

(i) $x_{r+1} = \frac{1}{2} \left(x_r + \frac{4}{x_r} \right)$ starting with $x_0 = 1$

(ii) $u_{r+1} = u_r + u_{r-1}$ starting with $u_1 = 1, u_2 = 3$

(Solutions on p. 49).

Exercise 3

Find the recurrence relation which describes the situation in which £20,000 is deposited initially in a savings account that yields 10% interest per annum and £4000 is withdrawn at the end of each year. When does the money run out?

(Solution on p. 49).

1.2 Classification of recurrence relations

The use of recurrence relations is usually fairly straightforward. However, their analysis is more difficult, except for certain special types of recurrence relation and we are therefore going to restrict ourselves to these special cases. Before we can begin we need to introduce four new terms into our vocabulary in order to be able to distinguish the differences and similarities between recurrence relations. For example how would we compare the recurrence relation

$$u_{r+1} = 3u_r + 2$$

and the recurrence relation

$$x_{r+1} = 4 - \frac{(x_r + 1)}{x_r^2}?$$

A **recurrence relation of k th order** is one where the difference between the highest and lowest subscripts is k .

The order of a recurrence relation is important because the sequence generated by a k th order recurrence relation requires k initial conditions.

In both the above examples the difference between the highest subscript ($r + 1$) and the lowest subscript (r) is $(r + 1) - r = 1$. Hence they are both *first order* recurrence relations.

A **linear recurrence relation** is one that can be written as

$$u_{r+1} = a_r u_r + b_r u_{r-1} + c_r u_{r-2} + \cdots + p_r$$

where p_r and the coefficients $a_r, b_r, c_r, \dots$ may depend on r , but do not depend on any of the u 's. Otherwise it is said to be **non-linear**.

The recurrence relations

$$u_{r+1} = 3u_r + 2$$

and

$$u_{r+1} = ru_r$$

are linear while the recurrence relation

$$x_{r+1} = 4 - \frac{x_r + 1}{x_r^2}$$

cannot be expressed in the above form and is thus non-linear.

A **linear constant-coefficient recurrence relation** is a linear recurrence relation where the coefficients $a_r, b_r, c_r, \dots$ are independent of r . (p_r may depend on r .)

This definition only applies to linear recurrence relations so we could not classify the formula

$$x_{r+1} = 4 - \frac{(x_r + 1)}{x_r^2}$$

in this way.

The recurrence relations

$$u_{r+1} = 3u_r + 2$$

and

$$u_{r+1} = 3u_r + 2r$$

have constant coefficients while the recurrence relation

$$u_{r+1} = ru_r$$

does not have constant coefficients because the coefficient of u_r depends on r .

A linear recurrence relation is **homogeneous** if $p_r = 0$ for all values of r . Otherwise it is said to be **nonhomogeneous**.

Again note that the definition only applies to linear recurrence relations. The recurrence relations

$$u_{r+1} = 3u_r$$

and

$$u_{r+1} = ru_r$$

are homogeneous while the recurrence relations

$$u_{r+1} = 3u_r + 2$$

and

$$u_{r+1} = ru_r + 3r$$

are nonhomogeneous.

Exercise 4
Classify the following recurrence relations by completing the table:

- (i) $u_{r+1} = 1.1u_r + 200$
- (ii) $x_{r+1} = 4 - \frac{(x_r + 1)}{x_r^2}$
- (iii) $t_{r+1} = 0.6t_r + 0.4t_{r-1}$
- (iv) $u_{r+1} = 4u_{r-1} + 7$
- (v) $u_r = 2ru_{r-1} - u_{r-2}$
- (vi) $y_r = 4y_{r-1}^2 + 3y_{r-2} + ry_{r-3}$
- (vii) $u_{r+1} = u_ru_{r-1}$

(Solution on p. 50).

order?	linear?	Constant-coefficient?	homogeneous?

1.3 The solution of linear first order recurrence relations

In certain simple cases it is possible to write down explicitly the general term in the sequence generated by a recurrence relation. The formula which does this is called the **solution of the recurrence relation**. In this subsection we shall examine how this can be done. The ability to write down solutions can be extremely useful in analysing the recurrence relation and it may also save some computation.

The computation of the first few terms of the sequence generated by the recurrence relation

u_{r+1} = 2u_r

starting with u_0 = 1 is fairly straightforward giving the following table of values.

r	0	1	2	3	4	5
u_r	1	2	4	8	16	32

However, if we required the value of u_{227}, the computation would be extremely tedious if done, for example, using the calculator. In some cases it is possible to find a formula which gives the values of the sequence without reference to the recurrence relation. In this example you can verify that the formula

u_n = 2^n

yields the same values as the above table. If this is true for all values of n, we can determine u_{227} as

u_{227} = 2^{227}

which is very easy to compute if you have a suitable calculator. This rather simple example can be generalized by considering the following questions:

- 1. What would happen if we started with another value for u_0, say u_0 = 5? If we start with u_0 = 5 in the recurrence relation

u_{r+1} = 2u_r

we obtain the following sequence.

r	0	1	2	3	4	5
u_r	5	10	20	40	80	160

This is exactly the same as before except that each term in the sequence has been multiplied by 5. Thus we can write down our solution as

u_n = 5 \times 2^n

The effect of changing the initial condition from u_0 = 1 to u_0 = 5 is to multiply all the terms in the sequence by 5.

- 2. Can we use these ideas to tackle the problem of solving the recurrence relation

u_{r+1} = 2u_r

with u_0 = A, where A is any number?

If we start with $u_0 = A$ and use the recurrence relation

$$u_{r+1} = 2u_r$$

to generate the sequence, we obtain the following table.

r	0	1	2	3	4	5
u_r	A	$A \times 2$	$A \times 4$	$A \times 8$	$A \times 16$	$A \times 32$

The pattern of the terms is emerging and we could confidently write down the general term

$$u_n = A \times 2^n \quad (n = 0, 1, 2, \dots)$$

By putting different values for A into this formula, we can arrive at the solution of the recurrence relation obtained earlier for the special initial conditions $u_0 = 1$ and $u_0 = 5$, and the solution for any other initial condition as well. A formula such as $u_n = A \times 2^n$ is called the **general solution** of the recurrence relation $u_{r+1} = 2u_r$; this distinguishes it from **particular solutions** such as $u_n = 2^n$ or $u_n = 5 \times 2^n$, which can be obtained from the general solution by giving particular numerical values, such as 1 or 5, to A . The particular value of A is determined by the initial condition.

Exercise 5

Write down the first five terms in the sequence for the recurrence relation

$$u_{r+1} = au_r$$

starting with the initial condition $u_0 = A$. Can you now write down the general solution for this recurrence relation?

(Solution on p. 50)

Exercise 6

Write down the general solution of the recurrence relation

$$u_{r+1} = 5u_r.$$

What is the particular solution if $u_0 = 3$?

(Solution on p. 50)

Exercise 7

(This is an important exercise which will be referred to in Section 3.)

Consider the recurrence relation

$$u_{r+1} = au_r$$

with $u_0 = 1$. For what values of a will the terms in the sequence

- (i) grow steadily
- (ii) diminish steadily
- (iii) oscillate but grow larger
- (iv) oscillate and diminish
- (v) stay the same size?

(Hint: Consider the following cases: $a > 1$, $a = 1$, $0 < a < 1$, $a = 0$, $-1 < a < 0$, $a = -1$, $a < -1$.)

(Solution on p. 50)

1.4 The nonhomogeneous linear constant-coefficient first order recurrence relation

We now turn to the slightly more difficult problem of finding the general solution of the recurrence relation

$$u_{r+1} = au_r + p$$

where a and p are given numbers which do not depend on r . This is the simplest kind of recurrence relation.

The most obvious way of trying to find the general solution is to start with $u_0 = A$, calculate the next few terms using the recurrence relation and to look for a pattern. The first few terms are

$$\begin{aligned}u_1 &= au_0 + p = aA + p \\u_2 &= au_1 + p = a(aA + p) + p = a^2A + ap + p \\u_3 &= au_2 + p = a(a^2A + ap + p) + p = a^3A + a^2p + ap + p \\u_4 &= au_3 + p = a(a^3A + a^2p + ap + p) + p \\&= a^4A + a^3p + a^2p + ap + p\end{aligned}$$

After only the first four terms we can guess that the solution is going to be

$$\begin{aligned}u_n &= a^nA + a^{n-1}p + a^{n-2}p + \cdots + ap + p \\&= a^nA + (a^{n-1} + a^{n-2} + \cdots + a + 1)p\end{aligned}$$

The expression in brackets is a geometric series. If we multiply it by $(a - 1)$ we get

$$\begin{aligned}(a - 1)(a^{n-1} + a^{n-2} + \cdots + a + 1) &= a^n + a^{n-1} + \cdots + a^2 + a \\&\quad - a^{n-1} - \cdots - a^2 - a - 1 \\&= a^n - 1\end{aligned}$$

(All terms cancel out except the first and last terms.)

So that, provided $a \neq 1$, we have

$$(a^{n-1} + a^{n-2} + \cdots + a + 1) = \frac{a^n - 1}{a - 1}$$

For $a = 1$ see Exercise 9.

Hence we can express the solution as

$$u_n = a^nA + \frac{(a^n - 1)}{(a - 1)}p$$

Finally, if we combine the terms in a^n , we have

$$u_n = \left(A + \frac{p}{a - 1}\right)a^n - \frac{p}{a - 1}$$

For the general solution of the recurrence relation we regard A as an arbitrary constant. A particular value for A gives a particular solution. This

last expression can be simplified further if we put $B = A + \frac{p}{a - 1}$ giving

$$u_n = Ba^n - \frac{p}{a - 1}$$

General solution of nonhomogeneous linear constant-coefficient first order recurrence relation.

where B is now regarded as the arbitrary constant which depends on the initial condition. Again a particular value for B gives a particular solution.

The simplest particular solution of all is the one for $B = 0$. It is

$$u_n = -\frac{p}{a - 1};$$

if we choose the right initial condition, all the terms of the sequence will be the same. For general values of B , the term $-p/(a - 1)$ still appears in the solution but there is now an additional term Ba^n (see the formula in the box above). In Section 1.3 we saw that this term (with A instead of B) was precisely the general solution of the homogeneous recurrence relation $u_{r+1} = au_r$. This new recurrence relation can be obtained from our original

(nonhomogeneous) recurrence relation $u_{r+1} = au_r + p$ by removing the nonhomogeneous term p (i.e. the term not containing any u). So the general solution shown in the previous box has the structure

$\left[\begin{array}{c} \text{general} \\ \text{solution} \end{array} \right] = \left[\begin{array}{c} \text{general} \\ \text{solution of} \\ \text{associated} \\ \text{homogeneous} \\ \text{problem} \end{array} \right] + \left[\begin{array}{c} \text{particular} \\ \text{solution} \end{array} \right]$	Structure of general solution
--	-------------------------------

This structure is important; it will recur in other contexts.

The structure we have just seen gives an alternative method for solving linear recurrence relations of the form

$$u_{r+1} = au_r + p$$

which can also be used for finding solutions of higher order recurrence relations. We look for a particular solution and add it to the general solution of the associated homogeneous recurrence relation

$$u_{r+1} = au_r.$$

Provided $a \neq 1$ this is easily done since there is a particular solution which is a constant.

Example 5

To find the general solution of the recurrence relation

$$u_{r+1} = 2u_r - 1$$

we first find the general solution of the associated homogeneous recurrence relation

$$u_{r+1} = 2u_r$$

obtained by dropping the nonhomogeneous term -1 from the original recurrence relation. This general solution is

$$u_n = B \times 2^n.$$

We now look for a particular solution of

$$u_{r+1} = 2u_r - 1$$

in the form $u_n = c$ where c is a constant. If such a solution exists it must satisfy the recurrence relation, i.e.

$$c = 2c - 1,$$

so that $c = 1$. Therefore

$$u_n = 1$$

is a particular solution.

Using the structure of the solution, the general solution of the original recurrence relation is therefore

$$u_n = B \times 2^n + 1.$$

Exercise 8

Find the general solutions of the following recurrence relations:

- (i) $u_{r+1} = 1.1u_r$
- (ii) $u_{r+1} = 1.1u_r + 200$
- (iii) $u_{r+1} = 3u_r + 8$

(Solutions on p. 51)

Exercise 9

What is the general solution of the recurrence relation

$$u_{r+1} = au_r + p$$

if $a = 1$? (Hint: Starting with $u_0 = A$ look for a pattern in the sequence $u_1, u_2, u_3, \dots$ in using the recurrence relation

$$u_{r+1} = u_r + p.)$$

(Solution on p. 51)

Important Exercise**1.5 Finding particular solutions**

Once we have found the general solution of any recurrence relation, we can pick out the particular solution appropriate to any given initial condition by using this condition to evaluate the arbitrary constant, B .

Example 6

To find the particular solution with $u_0 = 3$ for the recurrence relation

$$u_{r+1} = 4u_r + 6$$

we substitute for a and p in the formula

$$u_n = Ba^n - \frac{p}{a-1}$$

to give

$$u_n = B \times 4^n - 2.$$

The particular solution can be found using the initial condition as

$$u_0 = B - 2 = 3$$

i.e. $B = 5$.

Hence the particular solution is

$$u_n = 5 \times 4^n - 2.$$

If, instead of u_0 , we were given an initial condition such as $u_2 = 30$ in the above example we should find the correct value for B from

$$u_2 = B \times 4^2 - 2 = 30$$

i.e. $B = 2$. The particular solution would now be

$$u_n = 2 \times 4^n - 2.$$

Exercise 10

Find the particular solutions of the following recurrence relations with the given initial condition

(i) $u_{r+1} = 1.1u_r, \quad u_0 = 1000$

(ii) $u_{r+1} = 1.1u_r + 200, \quad u_1 = 1300$

(iii) $u_{r+1} = 3u_r + 8, \quad u_2 = 5$

(Solution on p. 51)

Exercise 11

Given that $u_n = \begin{cases} Ba^n - \frac{p}{a-1} & \text{if } a \neq 1 \\ A + np & \text{if } a = 1 \end{cases}$

express u_n as a function of u_0 , a , n and p .

(Solution on p. 51)

Summary of Section 1

1. An equation relating a term in a sequence to one or more previous terms, which can be used to generate the whole sequence, is called a **recurrence relation**.
2. A recurrence relation is of **kth order** if the difference between the highest and lowest subscript in it is k . The sequence generated by a k th order recurrence relation requires k **initial conditions**.
3. A recurrence relation is **linear** if it can be written as

$$u_{r+1} = a_r u_r + b_r u_{r-1} + c_r u_{r-2} + \cdots + p_r$$

where p_r and the coefficients $a_r, b_r, c_r, \dots$ may depend on r , but do not depend on any of the u 's. Otherwise it is said to be **non-linear**.

4. A linear recurrence relation is a **linear constant-coefficient recurrence relation** if none of the coefficients a_r, b_r, c_r depend on r . (p_r may depend on r .)
5. A linear recurrence relation is **homogeneous** if $p_r = 0$ for all values of r . Otherwise it is said to be **nonhomogeneous**.
6. The **general solution** of a linear first order constant-coefficient recurrence relation

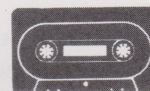
$$u_{r+1} = au_r + p$$

$$\text{is } u_n = \begin{cases} Ba^n - \frac{p}{a-1} & \text{if } a \neq 1 \\ A + np & \text{if } a = 1 \end{cases}$$

where A and B are arbitrary constants.

7. A **particular solution** for the recurrence relation is one in which the arbitrary constant is given a particular value.

2 Linear second order recurrence relations (Tape Section)



In the last section we gave an example of a linear second order recurrence relation when we discussed Fibonacci sequences in Example 3. The study of second order recurrence relations is probably more stimulating than first order relations because they give rise to more interesting sequences. The results of this section will also be useful when you study the units on second order differential equations and the numerical solution of ordinary differential equations.

2.1 An example

We begin with a modelling example taken from an economics textbook.

Example 1

(Do not spend too much time on the modelling process in this example.)

We wish to examine the demands for the installation of telephone lines in a certain city. We define the following variables:

Let p_r = price for the installation of a telephone in year r
 q_r = number of telephones installed in year r .

To produce a simple model we ignore, for example, the fact that there is a delay in installing the 'phone and we also ignore the effects of inflation. We also make the following assumptions:

- (i) If the price goes up there will be less demand for telephones. Suppose that, unknown to Telecom, the demand is given by a linear relationship of the form

$$\left[\begin{array}{l} \text{number of telephones} \\ \text{installed in year } r \end{array} \right] = 2000 - 3 \times \left[\begin{array}{l} \text{price of installation} \\ \text{in year } r \end{array} \right]$$

i.e. $q_r = 2000 - 3p_r$ (1)

- (ii) If sales go up there will be more pressure on Telecom engineers who have to do the installation. To control this demand Telecom have to fix the price of installation, p_{r+1} , at the beginning of year $r + 1$. We suppose that the method used by Telecom to determine the increase in price is to make the increase equal to 10% of the expected increase in demand for telephones in that year.

The price increase in year $r + 1$ is $p_{r+1} - p_r$.

The latest figures for the demand at the beginning of year $r + 1$ will be q_r and q_{r-1} . Hence the latest estimate of the increase in demand per year is $q_r - q_{r-1}$. Taking the price increase to be 10% of this we have

$$p_{r+1} - p_r = 0.1(q_r - q_{r-1}) \quad (2)$$

From Equation (1) we can see that

$$q_r - q_{r-1} = -3(p_r - p_{r-1})$$

and hence we can substitute this into Equation (2) to obtain the linear second order recurrence relation

$$\begin{aligned} p_{r+1} - p_r &= 0.1(q_r - q_{r-1}) \\ &= -0.3(p_r - p_{r-1}) \end{aligned}$$

$$\text{i.e. } p_{r+1} = 0.7p_r + 0.3p_{r-1}.$$

In order to start the computation we need two initial conditions. Suppose that p_0 , the price in year 0, is £100 and p_1 , the price in year 1, is £120; then we can determine the prices and sales in the following years using the recurrence relation.

$$\begin{aligned} \text{We have } p_2 &= 0.7p_1 + 0.3p_0 \\ &= 0.7 \times 120 + 0.3 \times 100 \\ &= 114 \\ p_3 &= 0.7p_2 + 0.3p_1 \\ &= 0.7 \times 114 + 0.3 \times 120 \\ &= 115.8 \end{aligned}$$

and so on.

The following table gives the prices and demands for telephones for the next few years.

Year: r	Price: p_r	No. of telephones: $q_r = 2000 - 3p_r$
0	100	1700
1	120	1640
2	114	1658
3	115.8	1653
4	115.26	1654
5	115.42	1654
6	115.37	1654

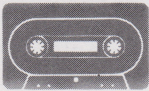
(Prices have been quoted to the nearest penny and demands to the nearest integer.)

The numbers in the table appear to be settling down to a constant value. By the time you have finished this section you should be able to determine the general solution for this problem and should thus be able to confirm the behaviour of the sequence.

One of the uses of this model could be to predict what might happen if, for some external reason, there was a sudden increase in price or if there were a sudden change, for example, in Telecom policy governing the method of determining the price increase in year $r + 1$.

You can see that second order recurrence relations are only a little harder to use than first order recurrence relations.

2.2 The solution of second order recurrence relations



In the rest of this section we look at the solutions of a simple type of second order recurrence relation: the linear constant-coefficient homogeneous relation

$$u_{r+1} = au_r + bu_{r-1}$$

where a and b are constants.

By using the tape along with the following frames you should be able to discover for yourself how to obtain general solutions of some recurrence relations of this type.

Now start the tape.

1

A linear, second order, homogeneous recurrence relation

$$u_{r+1} = 12u_r - 20u_{r-1}$$

particular
solutions

u_0	u_1	u_2	u_3	u_4	u_5	u_6
1	2					
1	10					
2	12					
5	26					

u_n

2

Solving $u_{r+1} = 12u_r - 20u_{r-1}$

Is the general solution
 $u_n = A2^n + B10^n$?

Try $u_n = Ax^n$

Divide by
 Ax^{r-1}

$$Ax^{r+1} = 12Ax^r - 20Ax^{r-1}$$

$$x^2 = 12x - 20$$

$$x = 2 \quad \text{or} \quad x = 10$$

$$u_n = A \times 2^n$$

or $u_n = B \times 10^n$

A and B will not
necessarily be
the same

Add solutions

$$u_n = A \times 2^n + B \times 10^n$$

$A = 0$ gives $u_n = B \times 10^n$
 $B = 0$ gives $u_n = A \times 2^n$

3

Rule for solving linear, second order, homogeneous recurrence relations

$$u_{r+1} = au_r + bu_{r-1}$$

STEP ONE : Form the auxiliary equation

$$x^2 = ax + b$$

STEP TWO : Solve the auxiliary equation

$$x = \lambda \text{ or } x = \mu$$

STEP THREE : Write down the general solution for the recurrence relation

$$u_n = A\lambda^n + B\mu^n$$

Watch out!

There are snags still to come...

4

Auxiliary equation with two equal roots

$$u_{r+1} = 4u_r - 4u_{r-1}$$

STEP ONE : Auxiliary equation is:

STEP TWO : Solution of auxiliary equation:

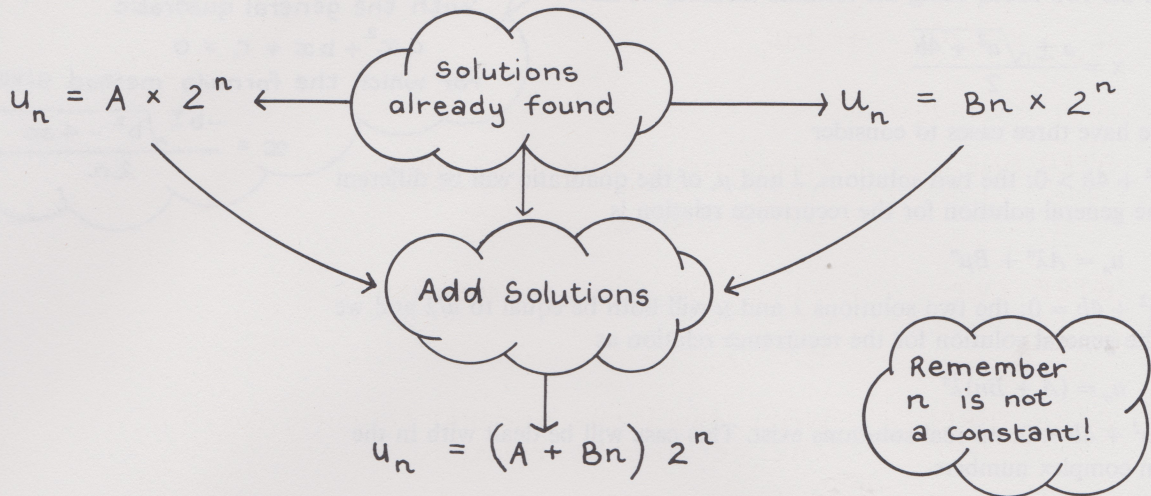
Particular
Solutions

u_0	u_1	u_2	u_3	u_4	u_5	u_6
1	2					
0	2					
0	2		3×2^3			

u_n

5

General solution of $u_{r+1} = 4u_r - 4u_{r-1}$



auxiliary equation

2 different roots ✓
2 equal roots ✓
no roots?

6

General rule

$$u_{r+1} = au_r + bu_{r-1}$$

STEP ONE:

Auxiliary equation

$$x^2 = ax + b$$

STEP TWO:

Solve auxiliary equation

STEP THREE:

different roots
 λ, μ

equal roots
 $\lambda = \mu$

no real roots

Solution is
 $u_n = A\lambda^n + B\mu^n$

Solution is
 $u_n = (A + Bn)\lambda^n$

Solution depends
on complex
numbers.
See Unit 5

We can go slightly further than we did on the tape by looking at the auxiliary equation in more detail. When we solve the auxiliary equation

$$x^2 - ax - b = 0$$

to find the two roots, using the formula method, we have

$$x = \frac{a \pm \sqrt{a^2 + 4b}}{2}$$

and we have three cases to consider

(i) $a^2 + 4b > 0$: the two solutions, λ and μ , of the quadratic will be different and the general solution for the recurrence relation is

$$u_n = A\lambda^n + B\mu^n$$

(ii) $a^2 + 4b = 0$: the two solutions λ and μ will both be equal to $a/2$ and we have the general solution for the recurrence relation as

$$u_n = (A + Bn)\lambda^n$$

(iii) $a^2 + 4b < 0$: no real solutions exist. This case will be dealt with in the unit on complex numbers.

Do not confuse this quadratic equation with the general quadratic

$$ax^2 + bx + c = 0$$

for which the formula method gives

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Exercise 1

Find the general solutions of the following linear homogeneous second order constant-coefficient recurrence relations

$$(i) \quad u_{r+1} = 7u_r - 12u_{r-1} \quad (iii) \quad u_{r+1} = u_r + u_{r-1}$$

$$(ii) \quad u_{r+1} = 6u_r - 9u_{r-1} \quad (iv) \quad u_{r+1} = 0.9u_r - 0.2u_{r-1}$$

(The third recurrence relation is the recurrence relation used to generate the Fibonacci Sequence and its solution will involve square roots.)

(Solution on p. 52)

2.3 Initial conditions

For a second order recurrence relation the general solution involves two arbitrary constants A and B . The two initial conditions required to start the sequence can also be used to determine the values of A and B required for the particular solution of the recurrence relation.

Example 2

Find the particular solution of the recurrence relation

$$u_{r+1} = 12u_r - 20u_{r-1}$$

with $u_0 = 5$ and $u_1 = 34$.

The general solution of this recurrence relation, which we found in the tape section, is

$$u_n = A \times 2^n + B \times 10^n$$

To determine the values of A and B we write down the equations for u_0 and u_1 in terms of A and B to obtain two equations. For this example we have

$$u_0 = A + B = 5 \quad (1)$$

$$u_1 = 2A + 10B = 34 \quad (2)$$

The solution to these equations can be found, for example, by subtracting twice the first equation from the second equation to give

$$8B = 24$$

$$\text{i.e.} \quad B = 3.$$

Substitution in equation (1) now gives

$$A + 3 = 5$$

$$\text{i.e.} \quad A = 2.$$

check in the second equation!

$$2 \times 2 + 10 \times 3 = 34$$

Hence the particular solution is

$$u_n = 2 \times 2^n + 3 \times 10^n$$

You may wonder whether it is always possible to find values for A and B given any initial conditions. Fortunately, it is possible to determine the arbitrary constants for any linear constant-coefficient homogeneous second order recurrence relation given the values of any two consecutive terms in the sequence.

Exercise 2

Find the particular solutions of the following recurrence relations satisfying the given initial conditions.

- (i) $u_{r+1} = 7u_r - 12u_{r-1}$ $u_0 = 2, u_1 = 7$
- (ii) $u_{r+1} = 6u_r - 9u_{r-1}$ $u_0 = 2, u_1 = 7$
- (iii) $u_{r+1} = u_r + u_{r-1}$ $u_0 = 1, u_1 = 1$
- (iv) $u_{r+1} = 0.9u_r - 0.2u_{r-1}$ $u_0 = 2, u_1 = 7$

(Solution on p. 52)

Exercise 3

For the general solution of the recurrence relation

$$u_{r+1} = u_r + u_{r-1}$$

given in Exercise 1(iii) what can you say about the behaviour of each part of the solution as n becomes large?

(Solution on p. 52)

Exercise 4

For the recurrence relation

$$u_{r+1} = \frac{3}{2}u_r + u_{r-1}$$

what happens to the term in the general solution as n becomes large? For the particular solution with $u_0 = 1, u_1 = -\frac{1}{2}$ what happens as n becomes large?

(Solution on p. 53)

Exercise 5

Find the general solution of the recurrence relation

$$p_{r+1} = 0.7p_r + 0.3p_{r-1}$$

used in modelling the telephones problem in Example 1. Find the particular solution if $p_0 = 100$ and $p_1 = 120$. What is the long-term behaviour of this particular solution?

(Solution on p. 53)

Summary of Section 2

- 1. Provided two initial conditions are given we can generate a sequence of numbers using a second order recurrence relation.
- 2. To obtain the general solution of the linear homogeneous constant-coefficient second order recurrence relation

$$u_{r+1} = au_r + bu_{r-1}$$

we first consider the auxiliary equation

$$x^2 = ax + b$$

There are 3 cases to consider

- (i) $a^2 + 4b > 0$: the auxiliary equation has two distinct solutions, λ and μ , and the general solution for the recurrence relation is

$$u_n = A\lambda^n + B\mu^n$$

Important Exercise

- (ii) $a^2 + 4b = 0$: the auxiliary equation has two equal solutions and the general solution is

$$u_n = (A + Bn)\lambda^n$$

- (iii) $a^2 + 4b < 0$: the auxiliary equation has no real solutions. (See Unit 5.)

3. To obtain a particular solution of the linear homogeneous constant-coefficient second order recurrence relation we use the two initial conditions to evaluate the arbitrary constants A and B in the general solution.

3 Numerical difficulties in using recurrence relations

3.1 Errors in computation and data

When the solution to a problem involves a lot of calculation, using either a calculator or a computer, we have to be aware that we are not going to get the exact solution to the problem. There are three distinct types of error that may affect the solution:

- (i) Rounding errors, due to the fact that numbers cannot be stored exactly, may introduce a significant error in the solution.
- (ii) The data given may not be exact.
- (iii) The mathematical formulation of a problem may involve approximations or simplifications.

In this section of the unit we shall only be looking at the problems which arise due to the first two types of error. We begin our discussion by describing two examples which illustrate some of the difficulties in using recurrence relations. The first is a very simple first order recurrence relation with constant coefficients while the second example is more realistic. By looking at the difficulties encountered using constant-coefficient recurrence relations you should see why the second problem causes trouble.

Example 1

Use the recurrence relation

$$u_{r+1} = 10u_r - 3$$

starting with $u_0 = \frac{1}{3}$ to evaluate u_{12} on your calculator. Compare your answer with the true solution.

The only difficulty I have is in storing $\frac{1}{3}$ in my calculator which displays 8 figures but actually calculates using 10 figures. Here are the displayed results I obtained.

r	u_r	r	u_r
0	0.333 333 33	6	0.333 3
1	0.333 333 33	7	0.333
2	0.333 333 33	8	0.33
3	0.333 333 3	9	0.3
4	0.333 333	10	0.
5	0.333 33	11	-3
		12	-33

You may get slightly different results if you use a different calculator.

The general solution for the recurrence relation, using the formula developed in Section 1, is

$$u_n = B \times 10^n + \frac{1}{3}$$

and the particular solution required is $u_n = \frac{1}{3}$. My computed solution is $u_{12} = -33$ whilst the theoretical solution is $u_{12} = \frac{1}{3}$. Hence my computed solution is totally wrong.

Example 2

The values of the integral

$$I_n = \int_0^1 x^n e^{x-1} dx$$

are required for various values of n . One way to do this is by numerical integration, given a particular value for n . However, integrating by parts once gives

$$\begin{aligned} I_n &= [x^n e^{x-1}]_0^1 - \int_0^1 nx^{n-1} e^{x-1} dx \\ &= 1 - n \int_0^1 x^{n-1} e^{x-1} dx \end{aligned}$$

But $\int_0^1 x^{n-1} e^{x-1} dx$ is I_{n-1} so that we obtain the recurrence relation (using r instead of n)

$$I_r = 1 - rI_{r-1}$$

What we have done is to replace the explicit definition of I_n , in a form we found difficult to handle, by a first order recurrence relation, which we can use to generate all the values for the integrals. Note that it is not a constant-coefficient recurrence relation.

To start the sequence we need an initial condition. We note that I_0 is given by

$$I_0 = \int_0^1 e^{x-1} dx = [e^{x-1}]_0^1 = 1 - e^{-1} = 0.632\,120\,558\,828\,557\dots$$

I used my calculator to compute the first 14 terms starting with $I_0 = 1 - \exp(-1) = 0.63212056$. The numbers produced appear in the following table.

r	I_r	r	I_r
0	0.63212056	8	0.10094291
1	0.36787944	9	0.09151379
2	0.26424112	10	0.08486208
3	0.20727665	11	0.06651712
4	0.17089342	12	0.20179456
5	0.14553291	13	-1.6233293
6	0.12680255	14	23.72661
7	0.11238214		

You may get slightly different results if you use a different calculator.

Now should we accept these results? Is the value of I_{14} approximately 23.7? To see that I have produced silly answers for I_{13} and I_{14} we return to the original definition of I_n as an integral. Since the value of I_n represents the area under the graph of $y_n = x^n e^{x-1}$ over the interval $[0, 1]$, it is useful to look at some of these graphs.

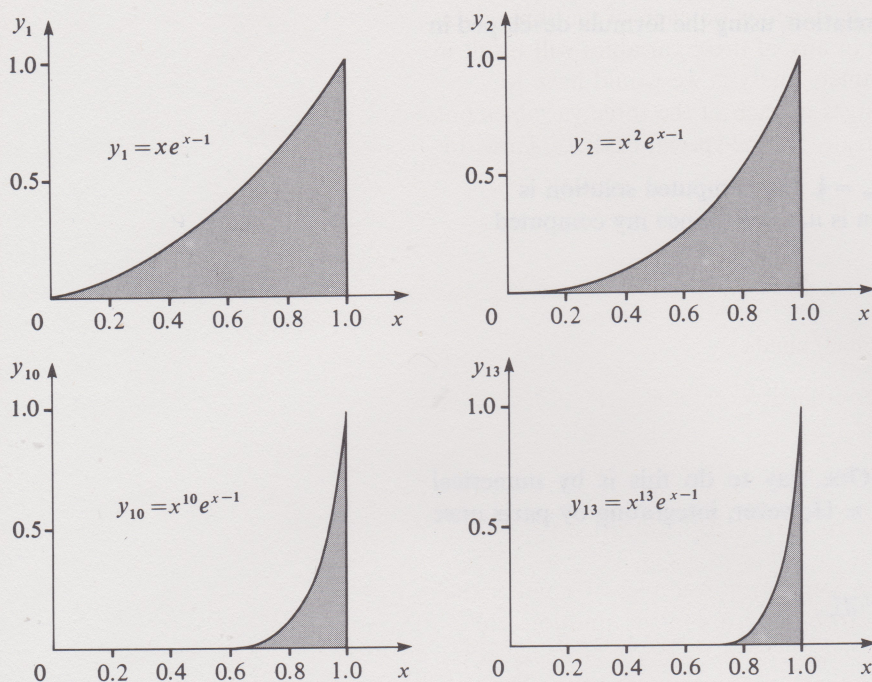


Figure 1

In each of the graphs you can see that the function we are integrating is positive in the interval $[0, 1]$ and we know that $x^{n+1} e^{x-1} \leq x^n e^{x-1}$ at any point in the interval $[0, 1]$. We conclude from this that the integral must be positive and also the values of the integrals should decrease as n increases. Our knowledge of the problem indicates to us that the values I obtained for I_{12} , I_{13} and I_{14} cannot be anywhere near the correct values.

This is a very interesting problem because we know how the solution should behave. For many practical problems we will have this insight and it can often tell us that things have gone wrong. The sole cause of the trouble in both the above examples is an apparently insignificant error in the initial condition. In both examples I made *no errors at all* in using the recurrence relation.

Both of the above examples illustrate a phenomenon called **ill-conditioning**. A problem is said to be **ill-conditioned** if a small change in the data for the problem results in a significant change in the solution. This definition is a bit vague because we need to know what the 'data' are and also what we mean by 'changes in the data' and 'significant changes in the solution'. We shall illustrate what this means by looking at the problem of finding u_N for a large value of N using the linear first order recurrence relation

$$u_{r+1} = au_r + p$$

with u_0 given and a and p known constants.

The data for the above problem are the values of u_0 , a and p . Each of them may be the result of some measurement or modelling process. As such they may be subject to measurement or modelling errors so that even before we start our computation we do not have exact information.

From Exercise 11 in Section 1 we can write down the particular solution for the above recurrence relation as

$$u_n = \begin{cases} \left(u_0 + \frac{p}{a-1}\right)a^n - \frac{p}{a-1} & \text{if } a \neq 1 \\ u_0 + np & \text{if } a = 1 \end{cases}$$

This formula tells us that the solution u_n depends on each of the three data variables u_0 , a and p so that a change in any of these variables will result in a change in the solution. To do a complete analysis we would have to determine how u_n varies for small changes in each of the three variables but, for simplicity, we are going to concentrate on one special case: i.e. How do small changes in u_0 affect u_n ?

Exercise 1

Find the solution, u_n , of

$$u_{r+1} = 10u_r - 3$$

given that $u_0 = \frac{1}{3} + \varepsilon$, where ε is a given (small) number.

Does a small change in u_0 lead to a smaller or larger change in u_{12} ?

(Solution on p. 53)

3.2 The effect of absolute errors in the initial condition

Examples 1 and 2 in the last subsection were both examples of small errors in the initial condition u_0 causing catastrophic errors in the solution. Here we assume that the coefficients in the recurrence relation are given exactly and that no rounding errors are made in any of the computations. The only error introduced will be an error in the initial condition.

Before we analyse the propagation of errors we shall briefly review the ways in which errors are measured. The error in an approximation $\bar{x}$ to a quantity whose exact value is x can be expressed in two ways:

- (i) The **absolute error** in the approximation is defined as $\bar{x} - x$
- (ii) The **relative error** in the approximation is defined as $\frac{\bar{x} - x}{x}$

It depends on the circumstances which is the appropriate measure of the error to use. In this subsection we examine the propagation of absolute errors while relative errors will be discussed in more detail in the next subsection.

Example 3

The value of π is to be approximated by 3.142. Since the correct value of π is 3.14159265... we can estimate the relative and absolute errors in the approximation.

The absolute error: $\bar{x} - x = 3.142 - 3.14159265 \dots$

$$= 0.00040734 \dots$$

$$\simeq 0.4073 \times 10^{-3}$$

The relative error: $\frac{\bar{x} - x}{x} = \frac{0.00040734 \dots}{3.14159265 \dots}$

$$= 0.0001296 \dots$$

$$\simeq 0.1296 \times 10^{-3}$$

To illustrate the way errors can build up in a recurrence relation calculation we go back to Example 1, which was to calculate u_{12} given that $u_0 = \frac{1}{3}$ and $u_{r+1} = 10u_r - 3$. Let us write $u_0, u_1, u_2, \dots$ for the solutions of the recurrence relation, starting from $u_0 = \frac{1}{3}$ and $\bar{u}_0, \bar{u}_1, \bar{u}_2, \dots$ for the solution starting from a slightly different value, $\bar{u}_0 = \frac{1}{3} + \varepsilon$. By Exercise 1, the solution starting from $\frac{1}{3} + \varepsilon$ is

$$\bar{u}_n = \frac{1}{3} + \varepsilon \times 10^n$$

How does the error in $\bar{u}_n$ depend on n ? With the initial condition $u_0 = \frac{1}{3}$ the n th term in the sequence is $u_n = \frac{1}{3}$. Hence the error in $\bar{u}_n$ is given by

$$\begin{aligned}\bar{u}_n - u_n &= \left(\frac{1}{3} + \varepsilon \times 10^n \right) - \frac{1}{3} \\ &= \varepsilon \times 10^n \\ &= (\bar{u}_0 - u_0) \times 10^n\end{aligned}$$

i.e. the absolute error in $\bar{u}_n$ is 10^n times the error in $\bar{u}_0$. We call the number 10^n the **scale factor** for the computation. We can see now what went wrong with the sequence generated in Example 1. By the time we compute $\bar{u}_{12}$ the error in $\bar{u}_0$ has been magnified 10^{12} times and even with a calculator which works to 10 figures any small absolute error in u_0 will swamp the solution.

Scale factors are treated in *M101* and *MS283*.

To describe such situations we introduce the term absolutely ill-conditioned, defined as follows.

A problem is said to be **absolutely ill-conditioned** if a small absolute error in the data gives rise to a significantly larger absolute error in the solution. If the absolute error in the solution is smaller than the absolute error in the data the problem is said to be **absolutely well-conditioned**.

In the example we have just considered, a small error ε in the data produces an error 10^{12} times as large in the solution, u_{12} , so this problem is absolutely ill-conditioned.

Using this definition of absolute ill-conditioning we shall now look at the general linear constant-coefficient first order recurrence relation

$$u_{r+1} = au_r + p$$

where the problem is to determine u_n given an initial value u_0 . If the approximation to u_0 is $\bar{u}_0 = u_0 + \varepsilon$, the approximations $\bar{u}_1, \bar{u}_2, \dots$ to the rest of the sequence will (assuming all calculations are done exactly) satisfy

$$\bar{u}_{r+1} = a\bar{u}_r + p$$

with the initial condition $\bar{u}_0 = u_0 + \varepsilon$.

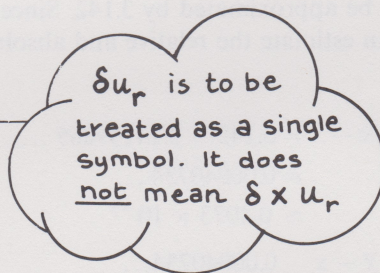
Subtracting the two recurrence relations we have

$$\bar{u}_{r+1} - u_{r+1} = a(\bar{u}_r - u_r).$$

Defining $\delta u_r = \bar{u}_r - u_r$ as the absolute error in u_r we obtain the homogeneous recurrence relation for δu_r as

$$\delta u_{r+1} = a \delta u_r$$

with $\delta u_0 = \bar{u}_0 - u_0 = \varepsilon$.



We have already examined the behaviour of this recurrence relation in Exercise 7 of Section 1 and we can summarize those results as

- (i) If $a > 1$ or $a < -1$, δu_n will grow in magnitude as n increases
- (ii) If $-1 \leq a \leq 1$, δu_n will not grow in magnitude as n increases.

We can also work out the scale factor for the computation of u_n since the solution of

$$\delta u_{r+1} = a \delta u_r$$

is
$$\begin{aligned}\delta u_n &= \bar{u}_n - u_n = a^n \delta u_0 \\ &= a^n (\bar{u}_0 - u_0).\end{aligned}$$

Hence the scale factor is a^n .

For large values of n we have the result that:

The problem of using the recurrence relation

$$u_{r+1} = au_r + p$$

to determine u_n for large values of n , given u_0 , is absolutely ill-conditioned with respect to small changes in u_0 if $|a| > 1$. If $|a| \leq 1$ the problem is well-conditioned.

Exercise 2

Determine the scale factors and comment on the absolute conditioning of the following problems

- (i) Find u_{20}
using $u_{r+1} = 3u_r + 2$
with $u_0 = 1$
- (ii) Find u_{20}
using $u_{r+1} = -\frac{1}{2}u_r + 6$
with $u_0 = 1$

(Solution on p. 53)

We can extend our analysis to Example 2 where we are using the recurrence relation

$$I_r = 1 - rI_{r-1}$$

to compute the values of the integrals using $I_0 = 1 - e^{-1}$. If we make a small absolute error in I_0 then the recurrence relation is

$$\bar{I}_r = 1 - r\bar{I}_{r-1}$$

Subtracting the two equations gives

$$\bar{I}_r - I_r = -r(\bar{I}_{r-1} - I_{r-1})$$

and putting $\delta I_r = \bar{I}_r - I_r$ to denote the error in I_r we have

$$\delta I_r = -r \delta I_{r-1}$$

Hence any absolute error in I_{r-1} will be magnified by $-r$ to give the error in I_r . To compute the scale factor for I_n we see that

$$\begin{aligned} \delta I_n &= -n \delta I_{n-1} \\ &= n(n-1) \delta I_{n-2} \quad (\text{Since } \delta I_{n-1} = -(n-1) \delta I_{n-2}) \\ &\vdots \\ &= (-1)^n n! \delta I_0 \end{aligned}$$

Hence the scale factor for this problem is $(-1)^n n!$. This becomes large very rapidly as n increases. For example if $n = 12$ then the scale factor is $12! = 479\,001\,600$. We can conclude that the problem in Example 2 is absolutely ill-conditioned.

Generally, the problem of determining u_n given u_0 using the recurrence relation

$$u_{r+1} = a_r u_r + p_r$$

will be absolutely ill-conditioned with respect to small changes in u_0 if $|a_r| > 1$ for all values of r .

Exercise 3

We wish to use the recurrence relation

$$u_{r+1} = ru_r - 1$$

with $u_0 = 1$, to compute u_n for large n . Determine the scale factor for the problem and comment on the absolute conditioning of the problem. Does the absolute conditioning depend on the initial condition u_0 ?

(Solution on p. 53)

Exercise 4

Use the recurrence relation

$$u_{r+1} = 10u_r - 3$$

to compute u_{10} for each of the three initial conditions

- (i) $u_0 = 1$
- (ii) $u_0 = 0.9999$
- (iii) $u_0 = 1.0001$

Comment on the answers obtained.

(Solution on p. 53)

3.3 The effect of relative errors in the initial condition

The results of the last exercise should have convinced you that absolute errors are not necessarily the best measure of the errors in the computation. If the terms in the sequence are also growing rapidly the growth of the absolute error may not have a *significant* effect on the value of u_n . Changing u_0 from 1 to 1.0001 changes the exact value of u_{10} from 6 666 666 667 to 6 667 666 667 (i.e. the solution has changed by one million). However the value of u_{10} is so large that the magnitude of the absolute error might be considered acceptable.

Look at what happens to the relative errors in this example. Writing $u_0 = 1$ and $\bar{u}_0 = 1.0001$ we have $u_{10} = 6\,666\,666\,667$ and $\bar{u}_{10} = 6\,667\,666\,667$.

The relative error in $\bar{u}_0$ is $\frac{\bar{u}_0 - u_0}{u_0} = \frac{0.0001}{1} = 0.1 \times 10^{-3}$

The relative error in $\bar{u}_{10}$ is $\frac{\bar{u}_{10} - u_{10}}{u_{10}} = \frac{1\,000\,000}{6\,666\,666\,667} = 0.00015$
 $= 0.15 \times 10^{-3}$

The magnitude of the relative error has grown by a factor of only 1.5 and, in situations where relative errors are more important than absolute errors, this might be deemed acceptable.

To emphasize the distinction between the growth of relative and absolute errors we define the term relatively ill-conditioned as follows:

A problem is said to be **relatively ill-conditioned** if a small relative error in the data gives rise to a significantly larger relative error in the solution. If the relative error in the solution is smaller than the relative error in the data the problem is said to be **relatively well-conditioned**.

Relative ill-conditioning occurs far less frequently than absolute ill-conditioning. Examples 1 and 2 in Subsection 3.1 were, however, illustrations of relative ill-conditioning. The feature which was common to both examples was that the scale factor was large while the solution we required was small. For first order recurrence relations these are precisely the conditions required for relative ill-conditioning.

To see how relative ill-conditioning can occur, we examine the problem of using the first order recurrence relation

$$u_{r+1} = au_r + p$$

to determine u_n given u_0 .

We have seen in Section 3.1 that the general solution can be expressed as

$$u_n = \begin{cases} \left(u_0 + \frac{p}{a-1}\right)a^n - \frac{p}{a-1} & \text{if } a \neq 1 \\ u_0 + np & \text{if } a = 1 \end{cases}$$

Now we know that *absolute* errors grow if $|a| > 1$. Suppose that we do have $|a| > 1$. Then u_n is given by

$$u_n = \left(u_0 + \frac{p}{a-1}\right)a^n - \frac{p}{a-1}$$

The first term on the right hand side is increasing rapidly in magnitude and normally the size of u_n will increase rapidly as n increases. However for one initial condition the value of u_n will remain small, i.e. when

$$u_0 = -\frac{p}{a-1}$$

in which case the solution is

$$u_n = -\frac{p}{a-1}$$

For the special initial condition $u_0 = -\frac{p}{a-1}$, therefore, the true solution remains small while the computed solution grows rapidly: the problem is relatively ill-conditioned. It turns out that relative ill-conditioning also occurs when u_0 is close to the special value $-\frac{p}{a-1}$ without being exactly equal to it. The analysis of this case will not be given here but you should know that it occurs.

These results can be summarized as follows:

The problem of determining u_n for sufficiently large n using the recurrence relation

$$u_{r+1} = au_r + p$$

is relatively ill-conditioned with respect to small changes in u_0 if

$$(i) \quad |a| > 1$$

and $(ii) \quad u_0 \simeq -\frac{p}{a-1}$

Exercise 5

For the recurrence relation

$$u_{r+1} = -3u_r + 2$$

show that the problem of determining u_{10} given $u_0 = \frac{1}{2}$ is relatively ill-conditioned, with respect to small changes in u_0 .

(Solution on p. 53)

Exercise 6

Find the values of u_{10} and $\bar{u}_{10}$ for the recurrence relation

$$u_{r+1} = \frac{1}{2}u_r + 3$$

with the initial conditions

- (i) $u_0 = 6$
- (ii) $\bar{u}_0 = 6 + \varepsilon$.

What are the absolute and relative errors in $\bar{u}_{10}$? Compare these with the absolute and relative errors in u_0 . Do your findings agree with the results on absolute and relative ill-conditioning?

(Solution on p. 54)

Exercise 7

- (i) Use your calculator to compute the next five terms in the sequence using the recurrence relation

$$u_{r+1} = 50u_r - 7$$

if the initial condition $u_0 = \frac{1}{7}$ is to be approximated by $\bar{u}_0 = 0.142857$.

- (ii) Compute the next five terms using the same recurrence relation if the initial condition is $u_0 = \frac{1}{6}$, approximated by $\bar{u}_0 = 0.166667$. Comment on the results of (i) and (ii).

(Solution on p. 54)

We can extend the above argument to non-constant-coefficient linear first order recurrence relations similar to the one satisfied by the integrals in Example 2. In that example, as in the constant-coefficient case, there is precisely one initial condition for which I_n remains small as n increases. Any other initial condition from $I_0 = 1 - e^{-1}$ will result in values of I_n which increase rapidly as n increases.

Generally, the problem of determining u_n for sufficiently large n , given the recurrence relation

$$u_{r+1} = a_r u_r + p_r$$

and the value of u_0 , will be relatively ill-conditioned, with respect to small changes in u_0 , if

- (i) $|a_r| > 1$ for all values of r
- (ii) u_0 is close to an initial condition for which the solution remains small (i.e. grows less rapidly than the solution of the associated homogeneous problem).

Exercise 8

In this question we look at the recurrence relation

$$u_{r+1} = 1 + (r+1)u_r$$

Compute the first 11 terms with

- (i) $u_0 = -1.7182818$
- (ii) $u_0 = -1.7182819$.

What conclusions can you draw from these results?

(Solution on p. 54)

Summary of Section 3

1. Errors arise in numerical calculations which may or may not severely affect the solution. The **absolute error** in the computed solution $\bar{u}_n$ compared with the true solution u_n is $\bar{u}_n - u_n$. The **relative error** in the computed solution $\bar{u}_n$ compared with the true solution u_n is given by $(\bar{u}_n - u_n)/u_n$.

2. A problem is said to be **absolutely ill-conditioned** if a small absolute error in the data gives rise to a significantly larger absolute error in the solution. A problem is said to be **relatively ill-conditioned** if a small relative error in the data gives rise to a significantly larger relative error in the solution. The growth of relative errors is particularly relevant when dealing with computations on the computer or calculator.

3. In this section we have concentrated on one of the more obvious cases in which relative and absolute ill-conditioning arises. Whilst relative ill-conditioning is a comparatively rare phenomenon you should be aware of the special cases in which it arises as described in this section.

4. For the recurrence relation

$$u_{r+1} = au_r + p$$

with a given initial condition u_0 the **scale factor** for the computation of u_n is a^n . If $|a^n|$ is significantly larger than 1 the problem of computing u_n is absolutely ill-conditioned with respect to small changes in u_0 . For sufficiently large n this occurs whenever $|a| > 1$. Similar results hold for the problem of computing u_n using the more general recurrence relation

$$u_{r+1} = a_r u_r + p_r$$

which is absolutely ill-conditioned with respect to small changes in u_0 , for sufficiently large n , if $|a_r| > 1$ for all values of r .

5. The computation of u_n using the recurrence relation

$$u_{r+1} = au_r + p$$

is relatively ill-conditioned with respect to small changes in u_0 for sufficiently large n if

- (i) $|a| > 1$
- (ii) $u_0 \simeq -\frac{p}{a-1}$

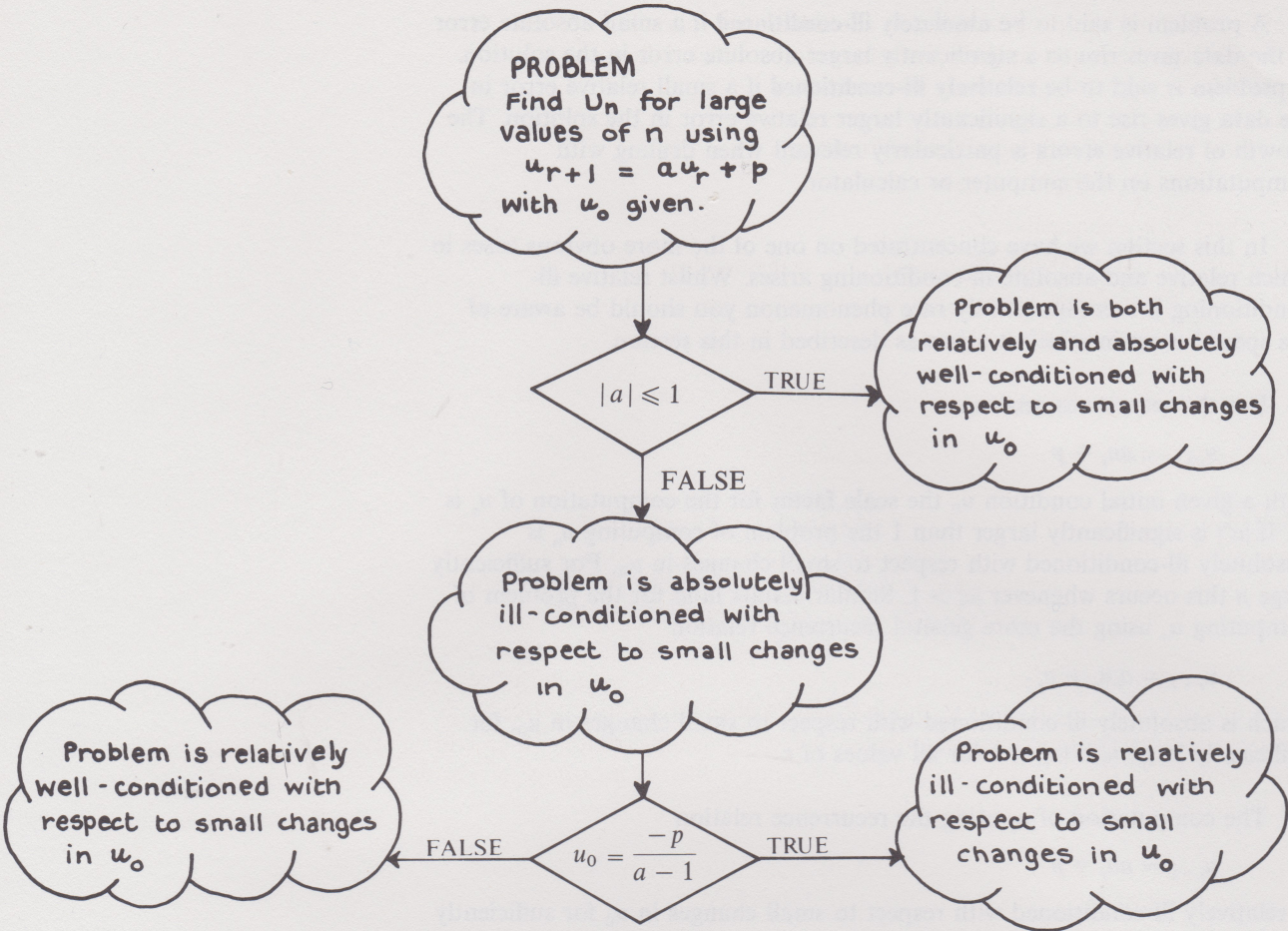
The problem of computing u_n using the more general recurrence relation

$$u_{r+1} = a_r u_r + p_r$$

is relatively ill-conditioned with respect to small changes in u_0 , for sufficiently large n , if

- (i) $|a_r| > 1$ for all r
- (ii) u_0 is close to an initial condition for which the solution remains small.

6. The following flowchart summarizes our findings for constant-coefficient linear first order recurrence relations.



4 Living with ill-conditioning

This section consists of two parts. The first is based on a television programme about a specific problem where ill-conditioning arises in a modelling context; the second (not based on the television programme) shows how to cope with ill-conditioning under certain circumstances.

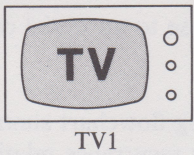
4.1 Modelling a mortgage (Television Subsection)

Read these notes before viewing the programme.

The story of the programme is a fairly straightforward analysis of the problem of determining the amount to repay each month after taking out a mortgage. It is not difficult to express this problem in a form which leads naturally to a recurrence relation.

However the purpose of the programme is not to teach you how to solve recurrence relations of this type. (This is done in Section 1 and you are asked to solve the recurrence relation, derived in the programme, as an exercise to be done before viewing the programme.) Rather the programme looks at two topics involved in the solution and these two topics form the two halves of the programme.

1. What are the modelling steps to be carried out in formulating the problem as a recurrence relation?
2. How does ill-conditioning show up in the solution of the mortgage problem?



Before watching the programme you should do the following exercise. The solutions can be found in the programme notes which follow.

Exercise

- (i) Find the general solution of the recurrence relation

$$u_{r+1} = 1.15u_r - 12M$$

where M is a constant.

- (ii) Given that $M = 258$, for what values of u_0 is the problem

- (a) absolutely ill-conditioned?
(b) relatively ill-conditioned?

Now watch the television programme 'Modelling a mortgage'.

Read these notes after viewing the programme.

One of the most important financial decisions in many people's lives is the decision whether or not to take out a mortgage to buy a house. In this programme we consider the case of a £20 000 loan which is to be paid back over 25 years when the interest rate is 15% per annum.

In accord with one of the basic principles of modelling the problem is kept as simple as possible, at least in the first stages of the modelling process. No account is taken of any extra payments, such as house insurance, which can be added on to the monthly repayment. We simply have a mortgage which is to be gradually repaid over a period of 25 years.

The way in which the building societies do their calculations is as follows. At the beginning of each year the 15% interest is added to the amount outstanding. During the following 12 months the repayments are made at monthly intervals. For example, the interest charged in the first year is 15% of £20 000 which is £3000. If M is the monthly repayment, the amount owed at the end of the first year is

$$20\,000 + 3000 - 12M$$

The building society will then charge interest in the second year, based on this amount. As this happens each year we can write down our basic equation as

$$\left[\begin{array}{c} \text{Amount owed} \\ \text{next year} \end{array} \right] = \left[\begin{array}{c} \text{Amount owed} \\ \text{this year} \end{array} \right] + [\text{interest}] - [\text{repayments}]$$

Suppose u_r represents the amount owed after r years. Our equation can be written as a recurrence relation:

$$u_{r+1} = u_r + \frac{15}{100}u_r - 12M$$

$$\text{i.e. } u_{r+1} = 1.15u_r - 12M \quad (1)$$

This linear, first order, nonhomogeneous recurrence relation is the one you solved before watching the programme. Its general solution (see the box on p. 13) is

$$u_n = B(1.15)^n + 80M \quad (2)$$

where B is an arbitrary constant.

This solution contains two unknowns B and M and we need two conditions to evaluate them. The two extra pieces of information which we have are:

- (i) The initial mortgage is £20 000, i.e. $u_0 = 20\,000$.
(ii) The mortgage is repaid in 25 years, i.e. $u_{25} = 0$.

Putting these values into the general solution (2) gives two simultaneous equations:

$$\begin{aligned} 20\,000 &= B + 80M \\ 0 &= B(1.15)^{25} + 80M \end{aligned}$$

Subtracting the second equation from the first we have

$$20\,000 = B(1 - (1.15)^{25})$$

and so

$$B = -626.58696 \dots$$

Substituting this value into the first equation gives

$$\begin{aligned} M &= \frac{20\,000 - B}{80} \\ &= 257.8324 \dots \end{aligned}$$

The monthly repayment figure M is not an amount we can pay exactly. The best we could do is to pay £257.83 per month or in whole pounds, as in the programme, £258. Using $M = 258$ we obtain the recurrence relation

$$u_{r+1} = 1.15u_r - 3096$$

This recurrence relation could now be used to determine how much you would owe each year. Rounding off each value of u_r , as it is calculated, to the nearest pound we obtain the following table of values:

r	u_r	r	u_r	r	u_r
0	20000	9	18389	18	12720
1	19904	10	18051	19	11532
2	19794	11	17663	20	10166
3	19667	12	17216	21	8595
4	19521	13	16702	22	6788
5	19353	14	16111	23	4710
6	19160	15	15432	24	2321
7	18938	16	14651	25	-427
8	18683	17	13753		

A rounding error of 17 pence per month plus the rounding of the interest payments to the nearest pound has resulted in the payment of £427 too much. Now 12 repayments of 17p is £2.04. Thus over 25 years you might have expected this to affect the result only by about £51. However, the pre-programme exercise should have prepared you for the magnification of this error. This problem is absolutely ill-conditioned for any value of u_0 since the factor 1.15 multiplying u_r in the recurrence relation is greater than one. Thus small absolute changes in the data can be expected to produce large absolute changes in the values of u_r .

In Section 3 only the effect of small changes in u_0 were examined. In the mortgage problem the small change was in the value of M . This illustrates the fact that a problem which is absolutely ill-conditioned will usually be sensitive to small changes in any of its data.

However, we do not discard this model even though it is absolutely ill-conditioned. The over-payment of £427 can easily be sorted out by the building society by adjusting the payments in the final year, for example. We do have a reasonable model to work with.

There were two other examples of how absolute changes in the data gave rise to large absolute changes in the solution.

- (i) Increasing the interest rate to 15.25% so that the coefficient of u_r changes from 1.15 to 1.1525 alters the value of u_{25} from -425 to 9833 .
- (ii) Increasing M from £258 to £263 alters u_{22} from 6788 to -1471 .

This last example means that paying £5 a month extra would result in approximately 40 fewer payments!

In the pre-programme exercise you should have calculated that the problem of using the recurrence relation to calculate the required values is relatively ill-conditioned for values of u_0 near £20 640. Since the starting value used is £20 000 this is close to the critical value. Hence we would expect small relative changes in the data to have a significant effect on the solution.

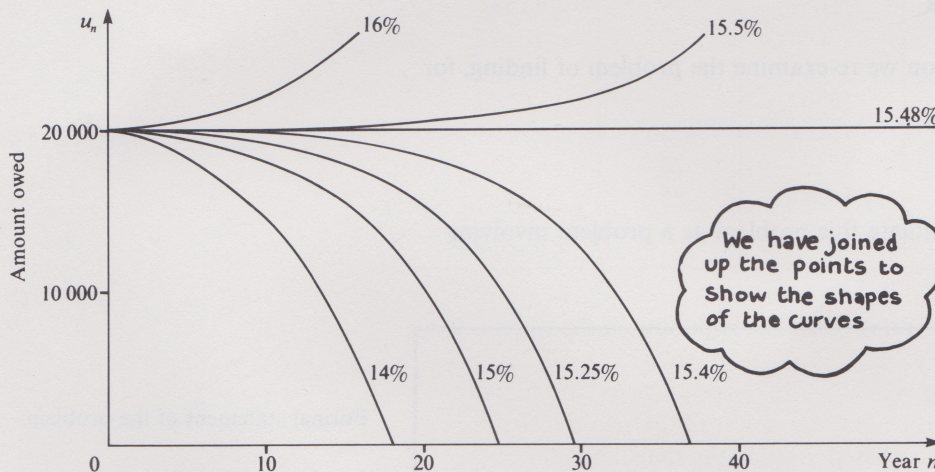
If you want to go into this a little further, notice that in the solution of the recurrence relation

$$u_{r+1} = au_r + p$$

given on p. 26 as

$$u_n = \left(u_0 + \frac{p}{a-1} \right) a^n - \frac{p}{a-1}$$

the behaviour of the solution (decreasing or increasing) depends on the sign of $\left(u_0 + \frac{p}{a-1} \right)$ so that if $u_0 \simeq -\frac{p}{a-1}$ a small change in the data (u_0 , p or a) can change a decreasing solution into an increasing solution. This qualitative change in the behaviour of the solution caused by the relative ill-conditioning shows up dramatically as we increase the interest rate by small amounts.



As we change the rate from 15.4% to 15.5% the solution changes from a decreasing solution to an increasing solution. The interest rate of 15.48% corresponds to the interest rate when we simply pay off the interest each year but never pay off any of the original loan. If the interest rate increases further the repayments are not sufficient even to pay off the interest and we end the year owing more than we did at the start of the year.

Exercise 1

What would be the monthly repayment for a £20 000 mortgage at 15% over

- (i) 15 years
- (ii) 20 years
- (iii) 30 years
- (iv) 35 years?

Comment on your results.

(Solution on p. 55)

Exercise 2

If the repayment on a £20 000 mortgage at 15 % is

- (i) £263
- (ii) £270

per month then how long does it take to pay off the mortgage?

Hint: To find n in the equation $a^n = b$ where a and b are known, we take logs of both sides to give $n \log a = \log b$ or $n = \frac{\log b}{\log a}$.

(Solution on p. 55)

Exercise 3

If the repayment on a £20 000 mortgage is fixed at £258 pounds, how long will it take to pay off the mortgage if the interest rate is

- (i) 12 %
- (ii) 13 %
- (iii) 14 %

(Solution on p. 55)

4.2 Coping with ill-conditioning

In the mortgage problem we saw an example of a problem which was ill-conditioned and yet the ill-conditioning could be tolerated because we knew how to live with the difficulty. In problems such as the integration problem in Section 3, Example 2 the ill-conditioning clearly cannot be lived with.

There is no method of obtaining satisfactory solutions to this problem. The way of overcoming the ill-conditioning here is to *change the problem*. If we can reformulate the problem as a well-conditioned problem we will be able to compute satisfactory solutions.

For the purposes of this discussion we re-examine the problem of finding, for example, the integral I_{12} where

$$I_n = \int_0^1 x^n e^{x-1} dx$$

Our first approach was to reformulate this problem as a problem involving recurrence relations as:

Find I_{12}
using $I_r = 1 - rI_{r-1}$
with $I_0 = 1 - e^{-1}$

Formal statement of the problem.

We saw that this problem is highly ill-conditioned (both relatively and absolutely) because a small change in the initial condition ‘wrecked’ the calculations.

Suppose that we were given a starting value other than a value for I_0 . For example, suppose we knew that $I_{20} = 0.0455$. We could utilize this information to work backwards to I_{12} by turning the recurrence relation around to give

$$I_{r-1} = \frac{1 - I_r}{r}$$

This is the same recurrence relation as above but we have now expressed I_{r-1} in terms of I_r . I could now reformulate the problem as

Find I_{12}
using $I_{r-1} = \frac{1 - I_r}{r}$
with $I_{20} = 0.0455$

Reformulation of the problem.

The method of working backwards using a recurrence relation is known, rather unsurprisingly, as **backward recurrence**. The following table shows my calculations

r	20	19	18	17	16	15
I_r	0.0455	0.047725	0.05011974	0.05277113	0.05571935	0.05901754

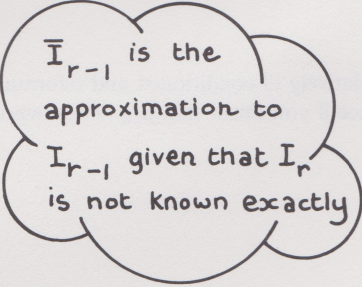
r	14	13	12
I_r	0.06273216	0.0669477	0.07177325

The first thing to notice about the above figures is how much more plausible they are than the results in Section 3. A rather more suprising discovery is that if you carry on working backwards to I_0 you will obtain the computed value $I_0 = 0.63212056$ which is correct to eight decimal places! This reformulation has given us a problem which is very well-conditioned indeed. The reason for this can be hinted at if we look at what happens if I_r is changed to $\bar{I}_r = I_r + \varepsilon$ where ε is a small error. If we substitute this into the recurrence relation

$$I_{r-1} = \frac{1 - I_r}{r}$$

we obtain

$$\begin{aligned}\bar{I}_{r-1} &= \frac{1 - \bar{I}_r}{r} \\ &= \frac{1 - I_r}{r} - \frac{\varepsilon}{r} \\ &= I_{r-1} - \frac{\varepsilon}{r}\end{aligned}$$



$\bar{I}_{r-1}$ is the approximation to I_{r-1} given that I_r is not known exactly

Hence the error in I_{r-1} caused by an error ε in I_r is

$$\bar{I}_{r-1} - I_{r-1} = -\frac{\varepsilon}{r}$$

Since r is greater than one, this error has been reduced. We can conclude that if we begin using I_{20} , for example, then, using backward recurrence, any error in I_{20} will be rapidly reduced as we compute I_{19} , I_{18} , I_{17} and so on. In fact, if the error in I_{20} is ε , the error in I_{19} is $-\varepsilon/20$ and the error in I_{18} is $-(-\varepsilon/20)/19 = \varepsilon/380$ and so on. By the time we have computed I_{14} the error ε in I_{20} has been reduced by a factor of 27 907 200 and would be fairly insignificant even if ε were large.

You may have accused me of cheating when I suddenly came out with the value 0.0455 for I_{20} as there was no obvious way of obtaining this value easily. However the above remarks indicate that we don't really need to know I_{20} very accurately in order to obtain accurate approximations for I_{12} . From our observations of the graphs of the integrands in Figure 1 of Section 3 we at least know that I_{20} lies between 0 and 1. The following table gives the sequence produced by the recurrence relation for each of these values of I_{20} . The values for $I_{20} = 0.0455$ are repeated for comparison.

r	$I_r(I_{20} = 0)$	$I_r(I_{20} = 0.0455)$	$I_r(I_{20} = 1)$
20	0	0.0455	1
19	0.05	0.047725	0
18	0.05	0.05011974	0.05263158
17	0.05277778	0.05277113	0.05263158
16	0.05571895	0.05571935	0.05572755
15	0.05901757	0.05901754	0.05901703
14	0.06273216	0.06273216	0.0627322
13	0.0669477	0.0669477	0.0669477
12	0.07177325	0.07177325	0.07177325

As you can see, whatever starting value we have for I_{20} between 0 and 1, the values for I_{13} and I_{12} will be the same correct to eight decimal places.

Can all ill-conditioned problems be reformulated as well-conditioned problems using backward recurrence? The answer is that whenever we have a problem of the form

$$u_{r+1} = a_r u_r + p_r$$

and $|a_r| > 1$ for all values of r , backward recurrence will help. This is because in backward recurrence we have

$$u_r = \frac{1}{a_r} u_{r+1} - \frac{p_r}{a_r}$$

and any error in u_{r+1} will be reduced in the computation of u_r . The only difficulty in using backward recurrence is in obtaining a suitable starting value.

Exercise 4

- (i) Show that the following problem is relatively ill-conditioned and reformulate it in a form suitable for backward recurrence if you know that u_{20} lies between 0 and 1.

Find u_{12}
using $u_{r+1} = 10u_r - 3$
with $u_0 = \frac{1}{3}$

- (ii) Use backward recurrence to solve for u_{12} for each of the following initial conditions:

- (a) $u_{20} = 1$
(b) $u_{20} = 0$
(c) $u_{20} = 100$

(Solution on p. 56)

Exercise 5

The general solution for the recurrence relation

$$u_{r+1} = 7u_r - 5r$$

is

$$u_n = B \times 7^n + \frac{5n}{6} + \frac{5}{36}.$$

- (i) Use your calculator to compute u_{15} , given that $u_0 = 5/36$.
 - (ii) Use backward recurrence to compute u_{15} , given that $u_{25} = 21$.
 - (iii) Compare your results (i) and (ii) with the correct values for u_{15} .
- (Solution on p. 56)

Exercise 6
The recurrence relation

$$u_{r+1} = 1 + r^2u_r$$

is to be used to generate a sequence of numbers. If it is known that the terms in the sequence tend to zero as r increases, determine an accurate value for u_1 by using backward recurrence with $u_{10} = 0$.

What is the calculated value of u_{10} if you start with your calculated value for u_1 as the initial condition?

(Solution on p. 56)

Summary of Section 4

1. The recurrence relation representing the mortgage problem, in which £ X is borrowed at $I\%$ interest and where the monthly repayment is £ M , is

$$u_{r+1} = \left(1 + \frac{I}{100}\right)u_r - 12M$$

The initial condition is $u_0 = X$.

Mortgage problems are often highly sensitive to small changes in data.

2. Ill-conditioning involving the first order recurrence relation

$$u_{r+1} = a_ru_r + p_r$$

with $|a_r| > 1$ for all values of r , can be cured by reformulating the problem as a problem involving **backward recurrence** using the recurrence relation

$$u_r = \frac{u_{r+1} - p_r}{a_r}$$

A suitable starting value must be obtained in order to use this recurrence relation.

5 Further exercises

Exercise 1
Classify the following recurrence relations by completing the table:

- (i) $u_{r+1} = ru_r$
- (ii) $x_{r+1} = \frac{1}{2}\left(x_r + \frac{7}{x_r}\right)$
- (iii) $u_{r+1} = 3u_{r-1} + r^2$
- (iv) $y_{r+1} = 3ry_r + 7y_{r-2}$
- (v) $z_{r+1} = z_r^2 + 2z_rz_{r-1}$

(Solution on p. 57)

order?	linear?	constant-coefficient	homo-geneous?

Exercise 2

Suppose that in a protected environment, in which there are no deaths (!), female seals reproduce according to the following rules.

- (i) Each productive female alive at the start of a year produces, on average, 0.75 new female seals in that year.
- (ii) A new born female seal becomes productive in her second year and is capable of producing new seals in each subsequent year.

Express this problem as a problem involving a recurrence relation, given that there are 100 female seals at the beginning of year 0 and 120 at the beginning of year 1.

(Solution on p. 57)

Exercise 3

Find the general solution for the recurrence relations

- (i) $u_{r+1} = 7u_r - 2$
- (ii) $u_{r+1} = u_r + 3.$

Find the particular solution of each recurrence relation with $u_0 = 3.$

(Solution on p. 57)

Exercise 4

Find the general solution of the linear homogeneous second order recurrence relations

- (i) $u_{r+1} = u_r + 0.75u_{r-1}$
- (ii) $u_{r+1} = 8u_r - 16u_{r-1}.$

Find the particular solutions for each of the recurrence relations for the initial conditions $u_0 = 100, u_1 = 120.$

(Solution on p. 57)

Exercise 5

Use your calculator to compute u_1, u_2, u_3, u_4 and u_5 using the recurrence relation

$$u_{r+1} = \frac{u_r}{(r+1)} + 1$$

- with (i) $u_0 = 1$
and (ii) $u_0 = 1.0001.$

Comment on your solution.

(Solution on p. 57)

Exercise 6

Use your calculator to compute u_1, u_2, u_3, u_4 and u_5 using the recurrence relation

$$u_{r+1} = (r+1)^2 u_r + 1$$

- with (i) $u_0 = -1.28$
(ii) $u_0 = -1.279.$

Comment on your solution.

(Solution on p. 58)

Exercise 7

Given that all the terms in the sequence $u_1, u_2, u_3, \dots$, generated by the recurrence relation in Exercise 6, lie between 0 and -1 , find u_5 correct to five decimal places.

(Solution on p. 58)

6 Computing

6.1 Computer packages

To help you to understand some of the numerical ideas and difficulties which arise from using the computer to solve mathematical problems we have developed computer packages which are to be used in conjunction with some of the units in this course. It is intended that, as part of your work for *Units*

1 and 2, you should make one visit to your nearest computer terminal to use the computer to explore problems which would be tedious and very time-consuming if tackled using a calculator.

The computer packages are computer programs, written by Student Computing Service programmers, which are designed so that you can tackle specific mathematical tasks. The detailed specifications of the individual packages can be found in the *Student Computing Service Student's Guide* check the supplementary material for notification of any modifications. None of the packages require any programming skills as they allow you to input and solve your problems by answering a series of questions posed by the package. The package will also attempt to help you if you get stuck.

The detailed instructions for logging on to the computer and using the packages can be found in the *Student Computing Service Student's Guide* which has been included with this mailing. If you are not familiar with the material in this guide you should read it before you make your terminal visit. Note that the packages are held in the program library for this course and you can obtain them by following the instructions for 'Using a Library Program' in the *Guide*.

When you have logged on, obtained the program and started to run it, the message

OPTION?

will be typed at your terminal. This indicates that you have to choose what to do next. Your response to the question OPTION? will usually be to

- (i) set up the problem or change it,
- (ii) select the method of solution,
- (iii) solve the problem or stop the program,
- (iv) request help if you are stuck.

After carrying out any task the program will return to this option point.

To show you how to use the program, we include an example of how to input a recurrence relation using the computer package RECREL. The detailed descriptions of the options will be found in *Unit 2*.

Example 1

We wish to look at the first ten terms in the sequence generated by the recurrence relation

$$u_{r+1} = 1 - (r + 1)u_r$$

with $u_0 = 0.63212056$. (This is the recurrence relation used in Example 2 of Section 3.) The following terminal dialogue shows how I input this problem, after logging in and obtaining the library program RECREL. Each of my responses is underlined to indicate that this is what I typed. The other characters in the dialogue were output by the computer.

Terminal Dialogue

OPTION? 10
 ORDER OF THE RECURRENCE RELATION (TYPE 1 OR 2)
 ORDER = ? 1

Comments

Option 10 enables you to tell the computer the order of the recurrence relation.
 The recurrence relation is of first order.

OPTION? 11
 INPUT THE RECURRENCE RELATION
 $U(R + 1) = ? 1 - (R + 1)*U(R)$

Option 11 enables you to tell the computer your recurrence relation.
 This stands for $u_{r+1} = 1 - (r + 1)u_r$ and should be input in this 'standard form' (see Subsection 6.3).

OPTION? 30
 INPUT INITIAL CONDITION
 $U(0) = ? 0.63212056$

Option 30 enables you to tell the computer the initial condition.
 The first term of the sequence must always be u_0 .

OPTION? 35
 INPUT NUMBER OF TERMS
 $N = ? 10$

Option 35 enables you to tell the computer how many terms in the sequence you wish to output.
 The first ten terms in the sequence are required.

OPTION? 20
 FORWARD RECURRENCE

Option 20 tells the computer the method you wish to use (forward recurrence).

OPTION? 41
 FULL PRINTOUT

Option 41 specifies what sort of printout you require (full printout).

OPTION? SOLVE

Option SOLVE tells the computer to perform the calculations.

The specification of each of the options is contained in *Unit 2*. The advantage of this method of input is that you can change any part of the problem without having to start again. For example, if you wish to look at the first fifteen terms in the sequence you would only have to change N using Option 35.

Notice that for this example I needed the following information:

- (i) the order of the recurrence relation
- (ii) the recurrence relation itself
- (iii) the initial condition
- (iv) the required number of terms in the sequence
- (v) the method to be used
- (vi) the type of printout required.

By looking at the list of options in *Unit 2* I discovered that the appropriate options were 10, 11, 30, 35, 20 and 41 respectively. It does not matter in which order I give this information to the program but it all has to be given before I use the option SOLVE. If you forget to input any of the information

when you try to SOLVE your problem, the program will tell you what information is missing and then return to the option point where you can input the missing data.

For each package there are five categories of option which can be used in response to the question OPTION?

(i) Command Options

The command options are specified by name. The following commands will be found in each of the packages

- OPTIONS —this tells the program to print out a list of the available options for the package
- SOLVE —this tells the program to run the problem
- HELP —this tells the program that you are stuck and need advice.
- LIST —this tells the program to list the information on the problem you have input.
- STOP —this is the only way to stop the program.

Other commands may be used in some packages.

(ii) Problem Options

Options numbered between 10 and 19 are used to submit problems or to change problems.

(iii) Method Options

Options numbered between 20 and 29 are used to choose the method used to solve the problem.

(iv) Data Options

Options numbered between 30 and 39 are used to submit data such as the number of terms, initial values and so on.

(v) Print Options

Options numbered between 40 and 49 are used to decide how much printout you require.

HELP

As stated earlier the command option HELP can be used to obtain guidance on how to proceed. However, if you want specific help with a particular option, which requires data to be input, you can ask for assistance within the option. Here is an example of the HELP command being used within an option.

Example 2

Suppose that you are not sure how to input a recurrence relation when using the package RECREL. To obtain help you would proceed as follows:

OPTION ? 11

INPUT THE RECURRENCE RELATION

$U(R + 1) = ?$ HELP

YOU SHOULD INPUT AN EXPRESSION IN STANDARD FORM TO REPRESENT THE RIGHT HAND SIDE OF THE RECURRENCE RELATION USING $U(R)$ TO DENOTE THE PREVIOUS TERM AND, IF NECESSARY, $U(R - 1)$ TO DENOTE THE PREVIOUS TERM BUT ONE.

AN EXPLANATION OF STANDARD FORMS CAN BE FOUND IN SECTION 6.3 OF UNIT 1. THE FOLLOWING ARE EXAMPLES OF VALID INPUTS.

$$U(R + 1) = ? 3 * U(R) - 4 * U(R - 1).$$

$$U(R + 1) = ? 3 * \text{SIN}(R) * U(R) + U(R - 1) \uparrow 2$$

INPUT YOUR RECURRENCE RELATION NOW

$$U(R + 1) = ?$$

6.2 The terminal visit

Before you go

Before you visit the terminal you should thoroughly prepare what you are going to do. You should determine

- (i) how to log on to the computer and run a library program. (This information can be found in the *Student Computing Service Student's Guide*.)
- (ii) what problems you wish to tackle. (A list of problems for *Unit 1* can be found in subsection 6.4.)
- (iii) how to input and solve each problem. (Check which options you need for each problem after reading Section 6 of *Unit 2*.)

If you have a clear idea of what you are going to do you will make optimum use of your time at the terminal.

The visit

When you make your visit to the terminal you should take with you a copy of the *Student Computing Service Student's Guide* plus the relevant units. For the package RECREL you will require *Units 1* and *2*. For each of the computing units there is a separate sheet which lists the available options. Take this with you and use it to find the correct option if you need to choose options while sitting at the terminal.

The postal service

If you find it impossible to visit a terminal a postal computing service is provided. This is not recommended except in extreme cases since the advantage of using the package yourself is that you can react to the output from the computer, changing the data, the method of solution and so on. Information on how to use this postal service can be found in the *Student Computing Service Student's Guide*

6.3 Valid input expressions

In several of the packages you will be asked to input an expression which contains variables, functions and arithmetic operators. There are rules for the input of expressions which ensure that the computer interprets the expression in an unambiguous way. These rules are almost identical to the rules governing the evaluation of an expression such as $x^2 + 3x + 9$, given a value of x , using the calculator.

One way to input the expression $x^2 + 3x + 9$ is to type in

$$X \uparrow 2 + 3 * X + 9$$

* stands for 'multiplied by'
 $\uparrow$ stands for 'raised to the power'

and this would be interpreted correctly by the computer for any value of X . This example illustrates some of the rules for arithmetic operators. Note that

the computer will evaluate $3 * X$ before adding it to $X \uparrow 2$. We say that multiplication has a higher **precedence** than addition. Similarly we would evaluate $X \uparrow 2$ first before doing any additions.

If we wished to input $x^{(2+y)}$ we would have to use brackets and type

$$X \uparrow (2 + Y)$$

to clarify the meaning.

Anything in brackets will be evaluated first so that

$$(X - Y) \uparrow 3 + (X + Y) \uparrow 4$$

stands for $(x - y)^3 + (x + y)^4$.

The order of precedence for the five arithmetic operators is as follows

level 1	() (brackets)
level 2	$\uparrow$ (raised to the power)
level 3	$*$ (multiplied by), $/$ (divided by)
level 4	$+$ (plus), $-$ (minus)

For example the computer evaluates an expression such as

$$(X + Y) \uparrow 3 * Z + 2 * X/Y$$

in several stages:

- (i) evaluate the expression inside the brackets first i.e. $(X + Y)$
- (ii) carry out the operations using $\uparrow$ i.e. $(X + Y) \uparrow 3$
- (iii) carry out all operations using $*$ and $/$ i.e. $(X + Y) \uparrow 3 * Z$ and $2 * X/Y$
- (iv) carry out all operations using $+$ and $-$ i.e. $(X + Y) \uparrow 3 * Z + 2 * X/Y$

Hence the expression represents

$$(x + y)^3 z + \frac{2x}{y}$$

There is a final rule needed to resolve the problems such as

$$2/Y * X$$

and

$$2 - Y + X.$$

The first could be interpreted either as $2/yx$ or $2x/y$ depending on the order of the operations carried out. Similarly the second expression could be interpreted either as $(2 - y) + x$ or $2 - (y + x)$. To avoid this ambiguity of operators at the same level of precedence we make the rule that operations at the same level are carried out from left to right.

Hence $2/Y * X$ represents $\frac{2}{y} \times x = \frac{2x}{y}$

and $2 - Y + X$ represents $(2 - y) + x$

Exercise 1

Interpret the following input expressions as algebraic expressions

- (i) $2 \uparrow (3 * X + Y) - X * Z/Y$
- (ii) $2 \uparrow X * 3 + 4 * X * Y - (3 + Z) \uparrow 2$

(Solution on p. 58)

Exercise 2

Express the following algebraic expressions as valid input expressions

(i) $(x + y)^3 + 3xy - \frac{y^x}{2}$

(ii) $x^3(y + 3)^2 - \frac{xy}{z} + \frac{z}{xy}$

(iii) $\frac{x + 3}{y + 7} + \frac{5 - x}{y^2}$

(Solution on p. 58)

Functions

There are several built-in functions which you can use when you input expressions. They consist of a three letter name followed by an expression in brackets. The following functions are allowed:

ABS the absolute value of the expression **ABS(X)** means $|x|$

e.g. **ABS(3)** = 3, **ABS(-3)** = 3

EXP the exponential function **EXP(X)** means e^x , $\exp(x)$

e.g. **EXP(3)** = e^3

INT the largest integer less than or equal to the value of the expression

e.g. **INT(4 · 3)** = 4, **INT(-4 · 3)** = -5

LOG the natural logarithm (to base e)

LOG(X) means $\log_e(x)$

SQR the positive square root of the expression

e.g. **SQR(4)** = 2

SIN the sine of the expression (in radians)

COS the cosine of the expression (in radians)

TAN the tangent of the expression (in radians)

ATN the arc tangent of the expression.

ATN(X) is in radians and will lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$

Here is an example of the functions being used

$$\text{SQR}(X \uparrow 2 + Y \uparrow 2) - \text{SIN}(X + \text{LOG}(Y)) - \text{TAN}(X)$$

The corresponding algebraic expression is

$$\sqrt{x^2 + y^2} - \sin(x + \log_e(y)) - \tan(x)$$

6.4 Computer exercises

Remember to plan your work carefully, as described in Subsection 6.2, before running these exercises at a terminal.

Exercise 3

Use forward and backward recurrence on the recurrence relation

$$u_{r+1} = 1 - (r + 1)u_r$$

with $u_0 = 1 - e^{-1} = 0.63212056 \dots$

which was used to find the values of the integrals discussed in Sections 3 and 4.

Compute the first 25 terms. Choose a suitable value for u_{35} for backward recurrence.

Exercise 4

The recurrence relation

$$u_{r+1} = (r+1)u_r - 1$$

with $u_0 = e - 1 = 1.7182818\dots$

can be used to evaluate the integrals

$$u_n = \int_0^1 x^n e^{1-x} dx$$

Use the package to compute the first 25 terms in this sequence.

Exercise 5

Check that your mortgage repayments are correct so that you repay the loan in the specified time period. (Note that your mortgage may include property and/or life insurance which are extras and should be omitted from the calculations.)

If you do not have a mortgage, work out the monthly payment for a £30 000 mortgage borrowed over 25 years if the interest rate is 17%. Use the package to determine the amount outstanding at the end of each year.

Exercise 6

Use the package to determine the behaviour of the sequences generated by the following second order recurrence relations.

(i) $u_{r+1} = u_r - u_{r-1}, \quad u_0 = 0, u_1 = 1$

(ii) $u_{r+1} = u_r - 2u_{r-1}, \quad u_0 = 0, u_1 = 1$

(iii) $u_{r+1} = 0.9u_r - 0.2u_{r-1} + 1500, \quad u_0 = 3600, u_1 = 4000$

(iv) $u_{r+1} = 8u_r - 16u_{r-1} + 4, \quad u_0 = \frac{4}{9}, u_1 = \frac{4}{9}$

Appendix: Solutions to the exercises

Solutions to the exercises in Section 1

1. (i) Since u_{r+1} is described in terms of u_r and u_{r-1} , it is a recurrence relation.

(ii) Only u_2 is described in terms of u_1 and we have no idea what u_3 is. This is not a recurrence relation.

(iii) This is a recurrence relation.

(iv) This is a recurrence relation because it can be used to generate a sequence. Note however that it is not expressed as a formula for u_{r+1} . If we wanted to we could add one to the subscript r , whenever it appears, to get

$$u_{r+1} = \frac{1}{(r+5)}u_r$$

and this would bring it into the usual form.

(v) This recurrence relation requires some manipulation to get it into a recognizable form. Reordering the equation gives

$$2u_{r+2} = 3u_{r+1} - u_r.$$

Dividing by two and reducing the subscripts by one gives

$$u_{r+1} = \frac{3}{2}u_r - \frac{1}{2}u_{r-1}.$$

2. The calculator gives the next five terms as follows.

(You may get slightly different results if you use a different calculator.)

(i) $x_1 = 2.5, x_2 = 2.05, x_3 = 2.006098, x_4 = 2.0000001, x_5 = 2$. (M101 and MS283 students will recognize, from Block 1, Unit 1, that the formula here is that used to find the square root.)

(ii) $u_3 = 4, u_4 = 7, u_5 = 11, u_6 = 18, u_7 = 29$.

3. Let u_r be the amount of money left in the account at the end of r years. In the following year the account increases by 10% and is subsequently reduced by the withdrawal of £4000. We express this mathematically as

$$\begin{aligned} u_{r+1} &= u_r + \frac{10}{100}u_r - 4000 \\ &= 1.1u_r - 4000 \end{aligned}$$

The fact that we initially have £20 000 in the account is given as $u_0 = 20000$.

Using the recurrence relation we obtain the following table

r	u_r
0	20000
1	18000
2	15800
3	13380
4	10718
5	7789.80
6	4568.78
7	1025.66

Since u_8 is negative there is not enough money in the account to be able to withdraw £4000 at the end of the eighth year. Hence the money runs out after eight years.

4.

	order?	linear?	constant-coefficient?	homo-geneous?
(i)	1	Yes	Yes	No
(ii)	1	No	N.A.	N.A.
(iii)	2	Yes	Yes	Yes
(iv)	2	Yes	Yes	No
(v)	2	Yes	No	Yes
(vi)	3	No	N.A.	N.A.
(vii)	2	No	N.A.	N.A.

(N.A. = ‘not applicable’, for these ideas are not defined for non-linear recurrence relations.)

5. The first five terms in the sequence are

r	u_r
0	A
1	$A \times a$
2	$A \times a^2$
3	$A \times a^3$
4	$A \times a^4$
5	$A \times a^5$

and we can immediately write down the general solution as

$$u_n = A \times a^n$$

6. By putting $a = 5$ in the formula derived in Exercise 5 we find that the general solution of

$$u_{r+1} = 5u_r$$

is

$$u_n = A \times 5^n$$

If $u_0 = 3$ then we have $A = 3$ giving the particular solution

$$u_n = 3 \times 5^n.$$

7. The particular solution can be written as

$$u_n = a^n.$$

From this we can make the following observations.

- (i) If $a > 1$ the solution grows exponentially.
- (ii) If $0 < a < 1$ the solution tends steadily to zero.
- (iii) If $a < -1$ the solution oscillates while increasing in size.
- (iv) If $0 > a > -1$ the solution oscillates and diminishes to zero.
- (v) The only cases we have not considered are those where a takes the values 1, 0 or -1 .

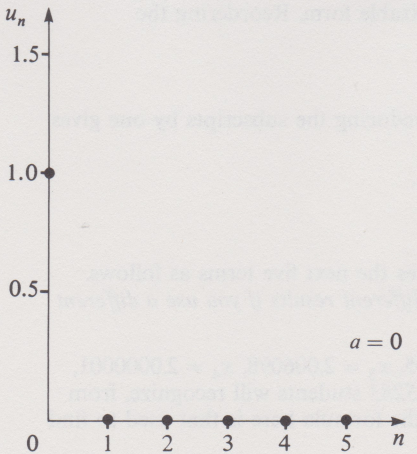
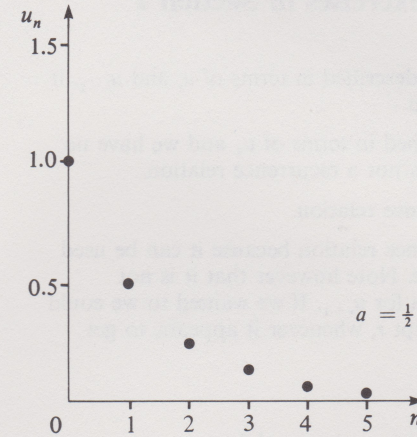
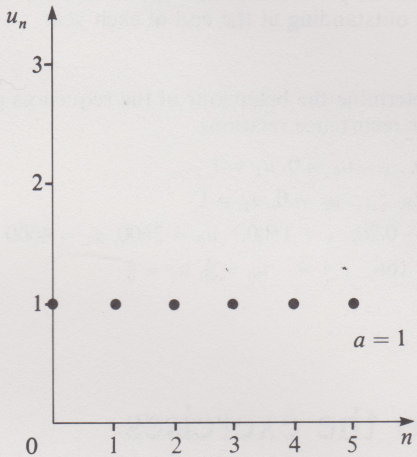
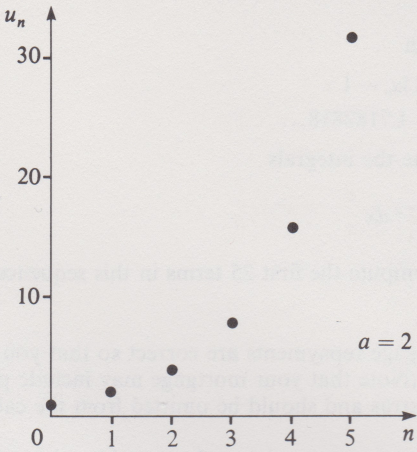
If $a = 1$ then $u_{r+1} = u_r$ and $u_n = 1$ for all values of n .

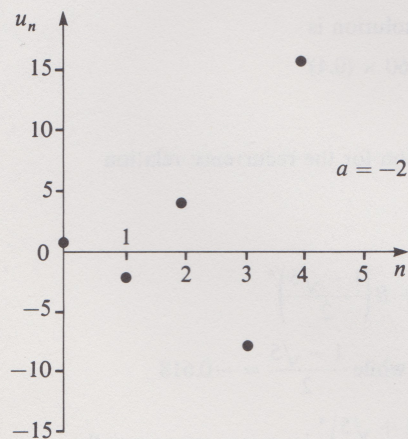
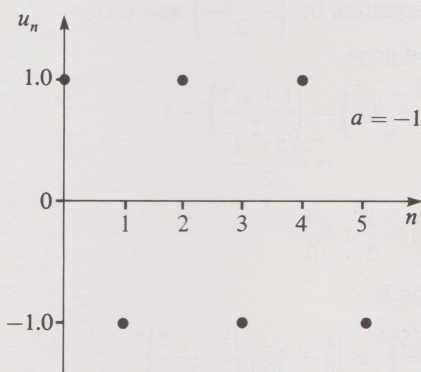
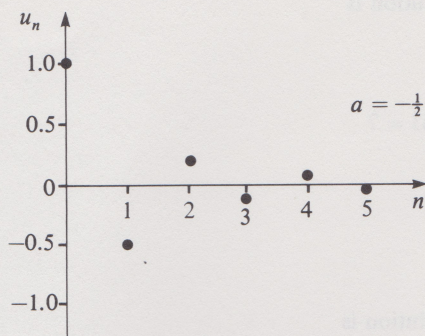
If $a = 0$ then $u_{r+1} = 0$ and $u_n = 0$ for all positive values of n .

If $a = -1$ then $u_{r+1} = -u_r$ and $u_{2n} = 1$, $u_{2n+1} = -1$ for all values of n .

In each of the above three cases the size of u_n does not change as n increases.

The following graphs illustrate the behaviour of the sequences generated by different values of a , with $u_0 = 1$.





8. (i) Using the form of the general solution given by

$$u_n = Ba^n - \frac{p}{a-1}$$

with $a = 1.1$ and $p = 0$ we have

$$u_n = B \times (1.1)^n$$

(ii) With $a = 1.1$ and $p = 200$ we have

$$u_n = B \times (1.1)^n - 2000$$

(iii) With $a = 3$ and $p = 8$ we have

$$u_n = B \times (3)^n - 4$$

9. If $a = 1$ we have

$$u_{r+1} = u_r + p$$

If we put $u_0 = A$ we obtain

$$u_1 = u_0 + p = A + p$$

$$u_2 = u_1 + p = (A + p) + p = A + 2p$$

$$u_3 = u_2 + p = (A + 2p) + p = A + 3p$$

$$u_4 = u_3 + p = (A + 3p) + p = A + 4p$$

and the general solution is

$$u_n = A + np.$$

10. Using the initial condition to evaluate B in the above general solution gives the particular solution

$$(i) \quad u_0 = 1000 = B \times (1.1)^0 = B$$

$$\therefore B = 1000$$

The particular solution is $u_n = 1000 \times (1.1)^n$

$$(ii) \quad u_1 = 1300 = 1.1B - 2000$$

$$1.1B = 3300$$

$$\text{i.e.} \quad B = 3000$$

The particular solution is

$$u_n = 3000 \times (1.1)^n - 2000$$

$$(iii) \quad u_2 = B \times 3^2 - 4 = 5$$

$$\text{i.e. } 9B = 9 \text{ giving } B = 1$$

The particular solution is

$$u_n = 3^n - 4$$

11. There are two cases to consider:

(i) $a \neq 1$

$$\text{We have } u_n = Ba^n - \frac{p}{a-1}.$$

$$\text{Now } u_0 = B - \frac{p}{a-1}$$

$$\text{hence } B = u_0 + \frac{p}{a-1}$$

$$\text{and } u_n = \left(u_0 + \frac{p}{a-1}\right)a^n - \frac{p}{a-1}.$$

(ii) $a = 1$

$$\text{We have } u_n = A + np.$$

Now $u_0 = A$

hence $A = u_0$

and $u_n = u_0 + np$.

Thus we can express the solution u_n as a function of u_0 , a , n and p as

$$u_n = \begin{cases} \left(u_0 + \frac{p}{a-1}\right)a^n - \frac{p}{a-1} & a \neq 1 \\ u_0 + np & a = 1 \end{cases}$$

Solutions to the exercises in Section 2

1. (i) The auxiliary equation is

$$x^2 = 7x - 12$$

which has solutions $x = 4$ and $x = 3$.

The general solution is

$$u_n = A \times 4^n + B \times 3^n$$

(ii) Here we have the special case in which we have equal solutions since the auxiliary equation

$$x^2 = 6x - 9$$

has solutions 3 and 3.

The general solution in this case is

$$u_n = (A + Bn)3^n$$

(iii) The auxiliary equation is

$$x^2 = x + 1$$

Using the formula for the solution of a quadratic, we have

$$x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

The general solution is

$$u_n = A \left(\frac{1+\sqrt{5}}{2}\right)^n + B \left(\frac{1-\sqrt{5}}{2}\right)^n$$

(iv) The auxiliary equation is

$$x^2 = 0.9x - 0.2$$

i.e. $x^2 - 0.9x + 0.2 = 0$

The formula method gives

$$x = \frac{0.9 \pm \sqrt{0.81 - 0.8}}{2} = \frac{0.9 \pm \sqrt{0.01}}{2} = \frac{0.9 \pm 0.1}{2}$$

Hence the two roots are 0.5 and 0.4 giving the general solution as

$$u_n = A(0.5)^n + B(0.4)^n$$

2. (i) The general solution is

$$u_n = A \times 4^n + B \times 3^n$$

$u_0 = 2$ gives $A + B = 2$

$u_1 = 7$ gives $4A + 3B = 7$

This has the solution $A = B = 1$

The particular solution is

$$u_n = 4^n + 3^n$$

(ii) The general solution is

$$u_n = (A + Bn)3^n$$

$u_0 = 2$ gives $A = 2$

$u_1 = 7$ gives $(A + B)3 = 7$

$$\text{or } B = \frac{1}{3}$$

$$u_n = \left(2 + \frac{n}{3}\right)3^n$$

(iii) The general solution is

$$u_n = A \left(\frac{1+\sqrt{5}}{2}\right)^n + B \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$u_0 = 1$ gives $A + B = 1$

$$u_1 = 1 \text{ gives } A \left(\frac{1+\sqrt{5}}{2}\right) + B \left(\frac{1-\sqrt{5}}{2}\right) = 1$$

Multiplying the first equation by $\left(\frac{1+\sqrt{5}}{2}\right)$ and then subtracting the second gives

$$B \left(\frac{1+\sqrt{5}}{2}\right) - B \left(\frac{1-\sqrt{5}}{2}\right) = \left(\frac{1+\sqrt{5}}{2}\right) - 1$$

$$B\sqrt{5} = \frac{\sqrt{5}-1}{2}$$

$$B = \frac{1}{2} - \frac{\sqrt{5}}{10} \text{ and } A = \frac{1}{2} + \frac{\sqrt{5}}{10}$$

The particular solution is

$$u_n = \left(\frac{1}{2} + \frac{\sqrt{5}}{10}\right) \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1}{2} - \frac{\sqrt{5}}{10}\right) \left(\frac{1-\sqrt{5}}{2}\right)^n$$

(iv) The general solution is

$$u_n = A(0.5)^n + B(0.4)^n$$

$u_0 = 2$ gives $A + B = 2$

$u_1 = 7$ gives $0.5A + 0.4B = 7$

This has solution $B = -60$, $A = 62$

Hence the particular solution is

$$u_n = 62 \times (0.5)^n - 60 \times (0.4)^n$$

3. The general solution for the recurrence relation

$$u_{r+1} = u_r + u_{r-1}$$

is given by

$$u_n = A \left(\frac{1+\sqrt{5}}{2}\right)^n + B \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$\text{Now } \frac{1+\sqrt{5}}{2} = 1.618 \text{ while } \frac{1-\sqrt{5}}{2} = -0.618$$

Thus as n increases $\left(\frac{1+\sqrt{5}}{2}\right)^n$ increases exponentially

while $\left(\frac{1-\sqrt{5}}{2}\right)^n$ oscillates and decreases. We can conclude that for large n we have

$$u_n = A \left(\frac{1+\sqrt{5}}{2}\right)^n \quad (n \text{ large})$$

For example, if we use the result for Exercise 2(iii) to compute u_6 then we have

$$\left(\frac{1}{2} + \frac{\sqrt{5}}{10}\right) \left(\frac{1+\sqrt{5}}{2}\right)^6 = 12.9846$$

while

$$\left(\frac{1}{2} - \frac{\sqrt{5}}{10}\right) \left(\frac{1 - \sqrt{5}}{2}\right)^6 = 0.0154$$

and already the term involving $\frac{1 + \sqrt{5}}{2}$ is the most significant part of the solution.

4. The auxiliary equation for this recurrence relation is

$$x^2 = \frac{3}{2}x + 1$$

i.e.

$$x^2 - \frac{3}{2}x - 1 = 0$$

The roots of this equation are 2 and $-\frac{1}{2}$ so that the general solution of the recurrence relation is

$$u_n = A \times 2^n + B \times \left(-\frac{1}{2}\right)^n$$

If you consider the two terms of the solution separately you can see that the term $A \times 2^n$ is doubling when you increase n by 1 while the term $B \times (-\frac{1}{2})^n$ is halving in size and oscillating. Thus as n increases the second term dies out and the solution is approximately

$$u_n = A \times 2^n \quad (n \text{ large})$$

There is one exception to this result and that is the case when $A = 0$. If the initial conditions are $u_0 = 1$, $u_1 = -\frac{1}{2}$ then the particular solution has $A = 0$ and $B = 1$ as

$$u_n = \left(-\frac{1}{2}\right)^n$$

Now as n becomes large the sequence tends to zero.

5. The auxiliary equation is

$$x^2 = 0.7x + 0.3$$

The roots of the auxiliary equation are 1 and -0.3 . Hence the general solution is

$$p_n = A + B \times (-0.3)^n$$

Using the initial conditions to determine the arbitrary constants A and B we have

$$p_0 = A + B = 100$$

$$p_1 = A - 0.3B = 120$$

Hence

$$1.3B = -20$$

i.e.

$$B = -15.3846... = -15.38 \text{ to the nearest penny}$$

and

$$A = 100 - B = 115.3846... = 115.38 \text{ to the nearest penny.}$$

This particular solution is

$$p_n = 115.38 - 15.38 \times (-0.3)^n$$

As n increases we can see that the second term in the solution diminishes rapidly and p_n tends to £115.38. After only six terms in Example 1 we had $p_6 = 115.37$.

Solutions to the exercises in Section 3

1. Using the formula

$$u_n = \left(u_0 + \frac{p}{a-1}\right) a^n - \frac{p}{a-1}$$

with $a = 10$, $p = -3$ and $u_0 = \frac{1}{3} + \varepsilon$ we have

$$\begin{aligned} u_n &= \left(\frac{1}{3} + \varepsilon\right) 10^n - \frac{(-3)}{10-1} \\ &= \varepsilon \times 10^n + \frac{1}{3} \end{aligned}$$

The change in the value of u_{12} is $\varepsilon \times 10^{12}$ which is much larger than the change (ε) in u_0 .

2. (i) The scale factor is a^n where $a = 3$ and $n = 20$.

Hence the scale factor is $3^{20} \approx 3 \times 10^9$

Thus the problem is absolutely ill-conditioned.

(ii) The scale factor is $(-\frac{1}{2})^{20} \approx 9 \times 10^{-7}$

Since this is very small the problem is absolutely well-conditioned.

3. If we start with

$$\bar{u}_{r+1} = r\bar{u}_r - 1$$

Subtracting the original recurrence relation gives

$$\begin{aligned} \bar{u}_{r+1} - u_{r+1} &= r(\bar{u}_r - u_r) \\ &= r(r-1)(\bar{u}_{r-1} - u_{r-1}) \quad \text{using } \bar{u}_r - u_r \\ &\vdots \\ &= r!(\bar{u}_0 - u_0) \end{aligned}$$

Putting $n = r + 1$ we have

$$\bar{u}_n - u_n = (n-1)!(\bar{u}_0 - u_0)$$

Hence the scale factor is $(n-1)!$ and the problem is absolutely ill-conditioned for large values of n . Since the scale factor does not depend on the initial condition u_0 absolute conditioning does not depend on u_0 . However, as you will see in Subsection 3.3, relative conditioning *does* depend on u_0 .

4. The exact values for u_{10} are

(i) 6 666 666 667

(ii) 6 665 666 667

(iii) 6 667 666 667

(You may get slightly different results on your calculator.)

Clearly the small changes in the data have produced very large absolute changes in the solution. For example changing the initial condition from $u_0 = 1$ to $u_0 = 1.0001$ has given rise to a difference of one million in the value of u_{10} . The problem is clearly absolutely ill-conditioned. However the fact that the first three figures in the solution are correct means that a small error in the initial condition does not really affect the *size* of the solution. This is discussed further in Subsection 3.3.

5. The general solution of the recurrence relation

$$u_{r+1} = -3u_r + 2$$

is

$$u_n = A \times (-3)^n + \frac{1}{2}$$

and the initial condition $u_0 = \frac{1}{2}$ implies that $A = 0$. Since the scale factor is $(-3)^{10}$ errors will grow rapidly and will quickly swamp the solution which is $u_{10} = \frac{1}{2}$. Hence the problem is relatively ill-conditioned.

To verify this we could put $\bar{u}_0 = \frac{1}{2} + \varepsilon$ in which case

$$\begin{aligned}\bar{u}_{10} &= (-3)^{10}\varepsilon + \frac{1}{2} = 59049\varepsilon + \frac{1}{2} \\ \frac{\bar{u}_{10} - u_{10}}{u_{10}} &= \frac{59049\varepsilon}{\frac{1}{2}} = 118098\varepsilon \\ \frac{\bar{u}_0 - u_0}{u_0} &= \frac{\varepsilon}{\frac{1}{2}} = 2\varepsilon\end{aligned}$$

Hence the relative error has been magnified by a factor of 59049.

6. With $u_0 = 6$ we have $u_{10} = 6$

With $\bar{u}_0 = 6 + \varepsilon$ we have $\bar{u}_{10} = 6 + \varepsilon \times (\frac{1}{2})^{10}$

Hence the absolute error in $\bar{u}_{10}$ is

$$\begin{aligned}\bar{u}_{10} - u_{10} &= \varepsilon \times \left(\frac{1}{2}\right)^{10} \\ &= 0.00097656\varepsilon\end{aligned}$$

The relative error in $\bar{u}_{10}$ is $\frac{\bar{u}_{10} - u_{10}}{u_{10}} = 0.00016276\varepsilon$

The absolute error in $\bar{u}_0$ is ε and the relative error is $\frac{\varepsilon}{6}$.

Both the absolute and relative errors have been reduced considerably and this agrees with the fact that the problem is both absolutely and relatively well-conditioned since $|a| < 1$.

7. Using the calculator with the given initial conditions gives the following sequences

(You may get slightly different results on your calculator.)

u_r	u_r (i)	u_r (ii)
0	0.142857	0.166667
1	0.14285	1.33335
2	0.1425	59.6675
3	0.125	2976.375
4	- 0.75	148811.75
5	-44.5	7440580.5

(i) The first problem is both relatively and absolutely ill-conditioned as the scale factor is $(50)^5 = 3.125 \times 10^8$ and the true solution is $u_5 = \frac{1}{2}$.

(ii) For the second problem the particular solution is

$$u_n = \frac{1}{42} \times (50)^n + \frac{1}{7}$$

and $u_5 = 7440476.3$

$$\begin{aligned}\frac{\bar{u}_5 - u_5}{u_5} &= \frac{104.2}{7440476.3} = 0.000014 = 1.4 \times 10^{-5} \\ \frac{\bar{u}_0 - u_0}{u_0} &= \frac{0.166667 - \frac{1}{6}}{\frac{1}{6}} = 0.000002 = 2 \times 10^{-6}\end{aligned}$$

The relative error has increased by a factor of 7 in the computation of $\bar{u}_5$ but this could well be an acceptable increase in the magnitude of the relative error. This agrees with the theoretical result that this problem is absolutely ill-conditioned. Although $\frac{1}{6}$ is ‘close to’ $\frac{1}{7}$, we may well consider that the problem of using the recurrence relation with $u_0 = \frac{1}{6}$ is *not* relatively ill-conditioned. This is really in the ‘grey area’ between being relatively ill-conditioned and relatively well-conditioned.

8. The first 11 terms were computed using the two initial conditions

(You may get slightly different results on your calculator.)

r	u_r (i)	u_r (ii)
0	-1.7182818	-1.7182818
1	-0.7182818	-0.7182819
2	-0.4365636	-0.4365638
3	-0.3096908	-0.3096914
4	-0.2387632	-0.2387656
5	-0.193816	-0.193828
6	-0.162896	-0.162968
7	-0.140272	-0.140776
8	-0.122176	-0.126208
9	-0.099584	-0.135872
10	0.00416	-0.35872
11	1.04576	-2.94592

This is a very interesting set of figures as you can see how the errors are building up. The last three terms in each of the sequences are totally different indicating that the problem of computing u_{11} with either of these two initial conditions is both relatively and absolutely ill-conditioned. We might also conclude that there is a value of u_0 in between -1.7182819 and -1.7182818 for which the solution remains small since the first sequence is beginning to increase rapidly while the second solution is beginning to decrease rapidly.

In fact the recurrence relation can be used to compute values for the integral

$$u_n = - \int_0^1 x^n e^{1-x} dx$$

for which the initial condition is
 $u_0 = 1 - e = -1.7182818 \dots$

Solutions to the exercises in Section 4

1. The general solution for the 15% mortgage where the repayment is £M per month is

$$u_n = B(1.15)^n + 80M$$

Since the initial condition is $u_0 = 20\,000$ the first equation in B and M will always be

$$B + 80M = 20\,000$$

The second equation depends on the length of the mortgage.

(i) For a 15 year mortgage we have $u_{15} = 0$

$$\text{i.e. } B(1.15)^{15} + 80M = 0$$

Solving the two simultaneous equations for B and M gives

$$B = \frac{20\,000}{1 - (1.15)^{15}} = -2802.27 \dots$$

$$M = \frac{20\,000 - B}{80} = 285.028 \dots$$

Hence the monthly repayment is £285.03

(ii) For a 20 year mortgage we have

$$B(1.15)^{20} + 80M = 0$$

$$B = \frac{20\,000}{1 - (1.15)^{20}} = -1301.529 \dots$$

$$M = £266.269 \dots$$

Hence the monthly repayment is £266.27.

(iii) For a 30 year mortgage we have

$$B = \frac{20\,000}{1 - (1.15)^{30}} = -306.69 \dots$$

$$M = 253.83 \dots$$

Hence the monthly repayment is £253.83.

(iv) For a 35 year mortgage we have

$$B = \frac{2000}{1 - (1.15)^{35}} = -151.31 \dots$$

$$M = 251.891 \dots$$

Hence the monthly repayment is £251.89.

The following table summarizes our results.

length of mortgage	monthly repayment
15	£285.03
20	£266.27
25	£257.83
30	£253.83
35	£251.89

It can be seen that there is a significant difference in the repayment between a 15 year and a 20 year mortgage but the difference approximately halves for each subsequent increase of 5 years. Notice how small the change is between a 25 year and a 35 year mortgage.

2 (i) If the repayment is £263 the recurrence relation for the 15% mortgage is

$$u_{r+1} = 1.15u_r - 12 \times 263$$

i.e.

$$u_{r+1} = 1.15u_r - 3156.$$

(One way of determining the repayment period is to use the recurrence relation to generate the figures. We present a different approach here.) The general solution is

$$\begin{aligned} u_n &= B(1.15)^n + \frac{3156}{1.15 - 1} \\ &= B(1.15)^n + 21040 \end{aligned}$$

Since the initial condition is $u_0 = 20\,000$

$$\text{we have } B + 21\,040 = 20\,000$$

$$\text{i.e. } B = -1040$$

Hence

$$u_n = -1040 \times (1.15)^n + 21\,040$$

We wish to know the value of n when $u_n = 0$.

i.e.

$$1040 (1.15)^n = 21\,040$$

$$(1.15)^n = \frac{21\,040}{1040}$$

and

$$n = \frac{\log \frac{21\,040}{1040}}{\log(1.15)} = 21.5166 \dots$$

take logs of both sides

Hence the mortgage is repaid after approximately $21\frac{1}{2}$ years. This agrees with the results obtained in the television programme.

(ii) For a repayment of £270 we have

$$u_{r+1} = 1.15u_r - 3240.$$

The general solution is

$$u_n = B(1.15)^n + 21\,600$$

and the initial condition $u_0 = 20\,000$ gives

$$B = -1600.$$

To find the repayment period n we have

$$1600 \times (1.15)^n = 21\,600$$

$$n = \frac{\log(13.5)}{\log(1.15)} = 18.622 \dots$$

Hence the repayment period is approximately $18\frac{1}{2}$ years.

3. One way of tackling this question is to use your calculator together with the recurrence relation

$$u_{r+1} = \left(1 + \frac{I}{100}\right)u_r - 3096$$

with $u_0 = 20\,000$ in order to obtain the appropriate values of n .

We are going to give an alternative method which leads to the same results.

(i) The recurrence relation for a 12% mortgage is

$$u_{r+1} = 1.12u_r - 3096$$

The general solution is

$$\begin{aligned} u_n &= B(1.12)^n + \frac{3096}{0.12} \\ &= B(1.12)^n + 25\,800. \end{aligned}$$

The initial condition $u_0 = 20\,000$ gives $B = -5800$. To get

the repayment period we need to determine the value of n for which $u_n = 0$ i.e.

$$5800(1.12)^n = 25\,800$$
$$\text{and } (1.12)^n = \frac{25\,800}{5800}$$

Taking logs of both sides gives

$$n = \frac{\log \frac{258}{58}}{\log 1.12} = 13.169 \dots$$

Hence the repayment period is approximately 13 years.

(ii) The recurrence relation is

$$u_{r+1} = 1.13u_r - 3096$$

The general solution is

$$u_n = B(1.13)^n + 23815.38$$

and the initial condition gives $B = -3815.38$.

In this case we have

$$n = \frac{\log \frac{23815.38}{3815.38}}{\log 1.13} = 14.9838.$$

The repayment period is approximately 15 years.

(iii) For a 14% mortgage we have

$$u_{r+1} = 1.14u_r - 3096.$$

The general solution is

$$u_n = B(1.14)^n + 22114.286$$

and the initial condition gives $B = -2114.286$

$$n = \frac{\log \left(\frac{22114.286}{2114.286} \right)}{\log 1.14} = 17.916 \dots$$

Hence the repayment period is approximately 18 years.

4. (i) The problem is relatively ill-conditioned as the coefficient of u_r is 10 and the initial condition u_0 is equal to the critical initial condition.

To use backward recurrence we have to guess the value of u_N for sufficiently large N . A reasonable estimate would, for example, be $u_{20} = 0$ since we know that any error in u_{20} will be reduced by a factor of 10 at each iteration. Formally we have

Find u_{12}
using $u_r = \frac{u_{r+1} + 3}{10}$
with $u_{20} = 0$.

(ii) The following table outlines the computations for each value of u_{20} .

r	$u_r(u_{20} = 1)$	$u_r(u_{20} = 0)$	$u_r(u_{20} = 100)$
20	1	0	100
19	0.4	0.3	10.3
18	0.34	0.33	1.33
17	0.334	0.333	0.433
16	0.3334	0.3333	0.3433
15	0.33334	0.33333	0.33433
14	0.333334	0.333333	0.333433
13	0.3333334	0.3333333	0.3333433
12	0.33333334	0.33333333	0.33333433

You can see that even with $u_{20} = 100$ we still have u_{12} correct to five decimal places.

5. (i)

r	u_r
0	0.13888899
1	0.97222222
2	1.80555556
3	2.63888889
4	3.4722221
5	4.305555
6	5.1388852
7	5.9721961
8	6.8053725
9	7.6376078
10	8.4632548
11	9.2427833
12	9.6994829
13	7.88963806
14	-9.7253356
15	-138.07735

(ii) For backward recurrence we use $u_r = \frac{u_{r+1} + 5r}{7}$

r	u_r
25	21
24	20.142857
23	19.306122
22	18.472303
21	17.6389
20	16.805557
19	15.972222
18	15.138889
17	14.305556
16	13.472222
15	12.638889

(iii) The particular solution we are looking for has $B = 0$ giving

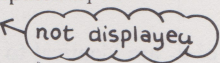
$$u_n = \frac{5n}{6} + \frac{5}{36}$$
$$\text{Hence } u_{15} = \frac{75}{6} + \frac{5}{36} = \frac{455}{36} = 12.638889$$

Backward recurrence, even with a small error in u_{25} , gives eight figures correct while the original formulation does not even give a solution with the correct sign.

6. We formulate the problem as

Find u_1
using $u_r = \frac{u_{r+1} - 1}{r^2}$
with $u_{10} = 0$

r	u_r
10	0
9	-0.01234568
8	-0.0158179
7	-0.02073098
6	-0.02835364
5	-0.04113415
4	-0.06507088
3	-0.11834121
2	-0.2795853
1	-1.2795853

On my calculator I computed u_1 as
 $u_1 = -1.2795853(023)$ 

Using this value for u_1 we have

r	u_r
1	-1.2795853
2	-0.2795853
3	-0.11834121
4	-0.06507088
5	-0.04113412
6	-0.02835312
7	-0.02071232
8	-0.01490368
9	-0.04616448
10	4.7393229

As you can see the errors build up slowly at first, but by the time we get to u_8 they are swamping the solution. The value for u_1 , computed using backward recurrence was correct to 10 significant figures.

Solutions to the exercises in Section 5

1.

order?	linear?	constant-coefficient?	homo-geneous
(i) 1	Yes	No	Yes
(ii) 1	No	N.A.	N.A.
(iii) 2	Yes	Yes	No
(iv) 3	Yes	No	Yes
(v) 2	No	N.A.	N.A.

2. We assume that no seals leave the environment either by dying or migrating. The basic equation is

number of female seals at start of year $r + 1$

$=$

number of female seals at start of year r

$+$

number of new born female seals in year r

Let S_r be the number of female seals at the beginning of year r . Then we have

$S_{r+1} = S_r +$

number of new born female seals in year r

The number of seals which are productive in year r is S_{r-1} and each seal has, on average, 0.75 female offspring. Hence we have

$S_{r+1} = S_r + 0.75S_{r-1}$

where $S_0 = 100$ and $S_1 = 120$.

3. The general solutions of the recurrence relations are

(i) $u_n = B \times 7^n + \frac{1}{3}$

(ii) $u_n = A + 3n$

For $u_0 = 3$ the particular solutions are

(i) $u_n = \frac{8}{3} \times 7^n + \frac{1}{3}$

(ii) $u_n = 3 + 3n$

4. (i) The auxiliary equation is $x^2 = x + 0.75$ and the two solutions are $x = \frac{3}{2}$ and $x = -\frac{1}{2}$. Hence the general solution is

$u_n = A \left(\frac{3}{2}\right)^n + B \left(-\frac{1}{2}\right)^n$

The particular solution is obtained by solving

$u_0 = A + B = 100$

$u_1 = \frac{3A}{2} - \frac{B}{2} = 120$

giving $A = 85$ and $B = 15$.

Hence the particular solution is

$u_n = 85 \times \left(\frac{3}{2}\right)^n + 15 \times \left(-\frac{1}{2}\right)^n.$

(ii) The auxiliary equation is $x^2 = 8x - 16$ and the two solutions are $x = 4$ and $x = 4$. Hence the general solution is

$u_n = (A + Bn)4^n.$

The particular solution is obtained by solving

$u_0 = A = 100$

$u_1 = (A + B)4 = 120$

giving $A = 100$ and $B = -70$.

Hence the particular solution is

$u_n = (100 - 70n)4^n.$

5. (You may get slightly different answers on your calculator.)

r	u_r (i)	u_r (ii)
1	1	1.0001
2	2	2.0001
3	2	2.00005
4	1.6666667	1.6666833
5	1.4166667	1.4166708
6	1.2833333	1.2833342

A small change in the value of u_0 has generated a different sequence. But note that the difference in the values for u_5 is 0.0000009 which is much less than the difference in the values for u_0 ($=0.0001$). This indicates that the problem is absolutely well-conditioned for small changes in the initial condition.

The relative change in u_0 of $\frac{0.0001}{1} = 0.0001$ has caused a relative change in u_5 of $\frac{0.0000009}{1.2833333} = 0.0000007$ which is

much smaller. This indicates that the problem is relatively well-conditioned for small relative changes in the initial condition. Both of these conclusions are consistent with the theory which indicates that if $|a_r| < 1$ for all values of r then the recurrence relation

$u_{r+1} = a_r u_r + p_r$ will be both relatively and absolutely well-conditioned with respect to small changes in u_0 .

6. (You may get slightly different results on your calculator.)

r	u_r (i)	u_r (ii)
1	-1.28	-1.279
2	-0.28	-0.279
3	-0.12	-0.116
4	-0.08	-0.044
5	-0.28	0.296
6	-6	8.4

From the tremendous change in the value of u_5 caused by a small change in u_0 we can see that the problem is both relatively and absolutely ill-conditioned for small changes in u_0 . This is consistent with the theory.

7. Using backward recurrence with the recurrence relation $u_r = \frac{u_{r+1} - 1}{(r + 1)^2}$ and starting with

(i) $u_{10} = 0$

(ii) $u_{10} = -1$

we obtain the following results:

(You may get slightly different results on your calculator.)

r	u_r (i)	u_r (ii)
1	0	-1
2	-0.01	-0.02
3	-0.01246914	-0.01259259
4	-0.01581983	-0.01582176
5	-0.02073102	-0.02073106
6	-0.02835364	-0.02835364

The two values of u_5 are identical to 8 decimal places and we could be extremely confident in quoting u_5 as

$u_5 = -0.02835$ (correct to 5 decimal places).

Solutions to the exercises in Section 6

1. (i) $2^{(3x+y)} - \frac{xz}{y}$
- (ii) $2^x \times 3 + 4xy - (3 + z)^2$
2. (i) $(X + Y) \uparrow 3 + 3 * X * Y - Y \uparrow X / 2$
- (ii) $X \uparrow 3 * (Y + 3) \uparrow 2 - X * Y / Z + Z / Y / X$
- (iii) $(X + 3) / (Y + 7) + (5 - X) / (Y \uparrow 2)$

There are variations of these answers which are also correct. For example, you may have $Z / (X * Y)$ or $Z / X / Y$ as the last term in (ii) and you may have omitted the brackets round $Y \uparrow 2$ at the end of (iii).

3, 4, 5, 6

We are not supplying answers to the computer exercises here. You will be able to obtain solutions at the terminal, as explained in Unit 2.

